On Hermite Hadamard-type Inequalities for Strongly log-convex Functions

Mehmet Zeki SARIKAYA, Hatice YALDIZ

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY
E-mail: sarikayamz@gmail.com, yaldizhatice@gmail.com

Article history: Received 28 November 2012, Accepted 25 February 2013, Published 1March 2013.

Abstract: In this paper, the notation of strongly log-convex functions with respect to $c > 0$ is introduced and versions of Hermite Hadamard-type inequalities for strongly logarithmic convex functions are established.

Keywords: Hermite-Hadamard's inequalities, log-convex functions, strongly convex with modulus $c > 0$.

Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important as described in the literature (see, e.g. [8], [3]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

The inequality (1.1) has evoked the interest of many mathematicians. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained, to mention a few, see ([1]-[12]) and the references cited therein.

Definition 1.1: The function $f : [a,b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds
\[ f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \]

for all \( x, y \in [a, b] \) and \( \lambda \in [0,1] \). We say that \( f \) is concave if \( (-f) \) is convex. In [6], Pearce et. al. generalized this inequality to \( r \)-convex positive function \( f \) which defined on an interval \([a, b]\), for all \( x, y \in [a, b] \) and \( t \in [0,1] \)

\[
f(tx + (1-t)y) \leq \begin{cases} \left[t[f(x)]^r + (1-t)[f(y)]^r\right]^{1/r}, & \text{if } r \neq 0 \\ \left[\frac{f(x)}{f(y)}\right]^{1-t}, & \text{if } r = 0. \end{cases}
\]

We have that 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions.

Recently, the generalizations of the Hermite-Hadamard's inequality to the integral power mean of a positive convex function on an interval \([a, b]\), and to that of a positive \( r \)-convex function on an interval \([a, b]\) are obtained by Pearce and Pecaric, and others (see [6]-[12]).

A function \( f : I \to [0, \infty) \) is said to be log-convex or multiplicatively convex if \( \log t \) is convex, or, equivalently, if for all \( x, y \in I \) and \( t \in [0,1] \) one has the inequality:

\[
f(tx + (1-t)y) \leq \left[ f(x) \right]^t \left[ f(y) \right]^{1-t}. \quad (1.2)
\]

We note that if \( f \) and \( g \) are convex and \( g \) is increasing, then \( g \circ f \) is convex; moreover, since \( f = \exp(\log f) \), it follows that a log-convex function is convex, but the converse may not necessarily be true [6]. This follows directly from (1.2) because, by the arithmetic-geometric mean inequality, we have

\[
\left[ f(x) \right]^t \left[ f(y) \right]^{1-t} \leq tf(x) + (1-t)f(y)
\]

for all \( x, y \in I \) and \( t \in [0,1] \).

For some results related to this classical results, (see [3], [4], [9], [10]) and the references therein. Dragomir and Mond [3] proved the following Hermite-Hadamard type inequalities for the log-convex functions:
\[ f\left(\frac{a + b}{2}\right) \leq \exp\left[ \frac{1}{b - a} \int_a^b \ln[f(x)] \, dx \right] \]
\[ \leq \frac{1}{b - a} \int_a^b G(f(x), f(a + b - x)) \, dx \]
\[ \leq \frac{1}{b - a} \int_a^b f(x) \, dx \]
\[ \leq L(f(a), f(b)) \]
\[ \leq \frac{f(a) + f(b)}{2}, \quad (1.3) \]

where \( G(p, q) = \sqrt{pq} \) is the geometric mean and \( L(p, q) = \frac{p - q}{\ln(p) - \ln(q)} \) (\( p \neq q \)) is the logarithmic mean of the positive real numbers \( p, q \) (for \( p = q \), we put \( L(p, q) = p \)).

Recall also that a function \( f : I \to \mathbb{R} \) is called strongly convex with modulus \( c > 0 \), if

\[ f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)(x - y)^2 \]

for all \( x, y \in I \) and \( t \in (0, 1) \). Strongly convex functions have been introduced by Polyak in [13] and they play an important role in optimization theory and mathematical economics. Various properties and applications of them can be found in the literature see ([13]-[16]) and the references cited therein.

In this paper we introduce the notation of strongly logarithmic convex with respect to \( c > 0 \) and versions of Hermite-Hadamard-type inequalities for strongly logarithmic convex with respect to \( c > 0 \) are presented. This result generalizes the Hermite-Hadamard-type inequalities obtained in [3] for log-convex functions with \( c = 0 \).

2. Main Results

We will say that a positive function \( f : I \to (0, \infty) \) is strongly log-convex with respect to \( c > 0 \) if

\[ f(\lambda x + (1 - \lambda)y) \leq \left[ f(x) \right]^\lambda \left[ f(y) \right]^{1-\lambda} - c\lambda(1 - \lambda)(x - y)^2 \]

for all \( x, y \in I \) and \( \lambda \in (0, 1) \). In particular, from the above definition, by the arithmetic-geometric mean inequality, we have

\[ f(\lambda x + (1 - \lambda)y) \leq \left[ f(x) \right]^\lambda \left[ f(y) \right]^{1-\lambda} - c\lambda(1 - \lambda)(x - y)^2 \]
\[ \leq \lambda f(x) + (1 - \lambda)f(y) - c\lambda(1 - \lambda)(x - y)^2 \]
\[ \leq \max\{f(x), f(y)\} - c\lambda(1 - \lambda)(x - y)^2 \quad (2.1) \]
Theorem 2.1 If a function $f : I \rightarrow (0, \infty)$ be a strongly log-convex with respect to $c > 0$ and Lebesgue integrable on $I$, we have

$$f\left(\frac{a+b}{2}\right) + \frac{c(b-a)^2}{12} \leq \frac{1}{b-a} \int_{a}^{b} G(f(x), f(a+b-x)) \, dx$$

$$\leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$

$$\leq L(f(a), f(b)) - \frac{c(b-a)^2}{6}$$

$$\leq \frac{f(a) + f(b)}{2} - \frac{c(b-a)^2}{6}$$

(2.2)

for all $a, b \in I$ with $a < b$.

Proof. From (2.1), we have

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2$$

$$\leq \lambda f(x) + (1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2.$$  

(2.3)

Since $f$ is a strongly log-convex function on $I$, we have for $x, y \in I$ with $\lambda = \frac{1}{2}$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y) - c(x-y)^2}{4}$$

$$\leq \frac{f(x) + f(y)}{2} - \frac{c(x-y)^2}{4}$$

(2.4)

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$f\left(\frac{a+b}{2}\right)$$

$$\leq \sqrt{f(ta + (1-t)b) f((1-t)a + tb)} - \frac{c(b-a)^2(1-2t)^2}{4}$$

$$\leq f(ta + (1-t)b) + f((1-t)a + tb) - \frac{c(b-a)^2(1-2t)^2}{4}.$$  

(2.5)

Integrating the inequality (2.5) with respect to $t$ over $(0,1)$, we obtain
\[ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) f(a+b-x) \, dx - \frac{c(b-a)^2}{12} \]
\[ \leq \frac{1}{b-a} \int_{a}^{b} A(f(x), f(a+b-x)) \, dx - \frac{c(b-a)^2}{12}, \]

and so for \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(a+b-x) \, dx, \)
\[ f\left(\frac{a+b}{2}\right) + \frac{c(b-a)^2}{12} \leq \frac{1}{b-a} \int_{a}^{b} G(f(x), f(a+b-x)) \, dx \]
\[ \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx. \] (2.6)

Since \( f \) is a strongly log-convex function on \( I, \) for \( x=a \) and \( y=b, \) we write
\[ f(tx + (1-t)y) \leq [f(a)]^{tx} [f(b)]^{1-ty} - c(1-y)(a-b)^2 \]
\[ \leq tf(a) + (1-t)f(b) - ct(1-y)(a-b)^2. \] (2.7)

Integrating the inequality (2.7) with respect to \( t \) over \( (0,1), \) we obtain,
\[ \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq f(b) \int_{0}^{1} \frac{f(a)}{f(b)} \, dt - c(b-a)^2 \int_{0}^{1} t(1-t) \, dt \]
\[ \leq f(a) \int_{0}^{1} t \, dt + f(b) \int_{0}^{1} (1-t) \, dt - c(b-a)^2 \int_{0}^{1} t(1-t) \, dt, \]
and so
\[ \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq L(f(a), f(b)) - \frac{c(b-a)^2}{6} \leq \frac{f(a) + f(b)}{2} - \frac{c(b-a)^2}{6}. \] (2.8)

Thus, from (2.6) and (2.8), we obtain the inequality of (2.2). This completes the proof.

**Theorem 2.2** Let a function \( f : I \to [0,\infty) \) be a strongly log-convex with respect to \( c > 0 \) and Lebesgue integrable on \( I, \) then the following inequality holds:
\[ \frac{1}{b-a} \int_{a}^{b} f(x) f(a+b-x) \, dx \leq f(a) f(b) + \frac{c^2(b-a)^4}{30} \]
\[ - \frac{4c(b-a)^2}{\ln(f(b) - f(a))} \left[ A(f(a), f(b)) + L(f(a), f(b)) \right] \] (2.9)
for all \( a, b \in I \) with \( a < b \).

**Proof.** Since \( f \) is strongly log-convex with respect to \( c > 0 \), we have that for all \( t \in (0,1) \)

\[
f(ta + (1-t)b) \leq [f(a)]^t [f(b)]^{1-t} - ct(1-t)(b-a)^2
\]

\[
\leq tf(a) + (1-t)f(b) - ct(1-t)(b-a)^2
\]

and

\[
f((1-t)a + tb) \leq [f(a)]^{1-t} [f(b)]^t - ct(1-t)(b-a)^2
\]

\[
\leq (1-t)f(a) + tf(b) - ct(1-t)(b-a)^2.
\]

Multiplying both sides of (2.10) by (2.11), it follows that

\[
f(ta + (1-t)b) f((1-t)a + tb) \leq f(a) f(b) + c^2 (b-a)^2 t^2 (1-t)^2
\]

\[
- c(b-a)^2 t (1-t) \left( f(b) \left[ \frac{f(a)}{f(b)} \right]^t + f(a) \left[ \frac{f(b)}{f(a)} \right]^t \right).
\]

(2.12)

Integrating the inequality (2.12) with respect to \( t \) over \( (0,1) \), we obtain

\[
\int_a^b f(ta + (1-t)b) f((1-t)a + tb) dt \leq \int_0^1 f(a) f(b) dt + c^2 (b-a)^2 \int_0^1 t^2 (1-t)^2 dt
\]

\[
- c(b-a)^2 \int_0^1 t (1-t) \left[ \frac{f(a)}{f(b)} \right]^t dt - c(b-a)^2 \int_0^1 t (1-t) \left[ \frac{f(b)}{f(a)} \right]^t dt
\]

\[
= \int_0^1 f(a) f(b) dt + c^2 (b-a)^2 \int_0^1 t^2 (1-t)^2 dt - c(b-a)^2 \int f(b) I_1 - c(b-a)^2 \int f(a) I_2.
\]

(2.13)

Integrating by parts for \( I_1 \) and \( I_2 \) integrals, we obtain
\[ I_1 = \int_0^1 t(1-t) \left[ \frac{f(a)}{f(b)} \right]' \, dt \]

\[ = t(1-t) \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ \frac{f(a)}{f(b)} \right]' \bigg|_0^1 - \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \int_0^1 (1-t) \left[ \frac{f(a)}{f(b)} \right]' \, dt \]

\[ = -\frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ (1-2t) \frac{1}{\ln \left[ \frac{f(a)}{f(b)} \right]} \left[ \frac{f(a)}{f(b)} \right]' \bigg|_0^1 + \frac{2}{\ln \left[ \frac{f(a)}{f(b)} \right]} \int_0^1 \left[ \frac{f(a)}{f(b)} \right]' \, dt \right] \]

\[ = \frac{1}{f(b)} \left[ \ln (f(a)-f(b)) \right]^2 + \frac{2f(a)-2f(b)}{\ln (f(a)-f(b))}, \tag{2.14} \]

and similarly we get,

\[ I_2 = \int_0^1 t(1-t) \left[ \frac{f(b)}{f(a)} \right]' \, dt \]

\[ = \frac{1}{f(a)} \left[ \ln (f(b)-f(a)) \right]^2 + \frac{2f(b)-2f(a)}{\ln (f(b)-f(a))}. \tag{2.15} \]

Putting (2.14) and (2.15) in (2.13), and if we change the variable \( x := ta + (1-t)b, \ t \in (0,1) \), we get the required inequality in (2.9). This proves the theorem.

**References**


