Degree Exponent Adjacency Eigenvalues and Energy of Specific Graphs

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Abstract: The Degree exponent DE matrix of a graph G, denoted by DE(G) whose vertex vi has degree di, is defined as the n × n matrix whose (i, j)-th entry is (di)ji, if vi and vj are adjacent and 0 for other cases. The Degree exponent energy (DEE) of G is the sum of absolute values of the eigenvalues of DE of G. In this paper, we present our results on degree exponent polynomial and degree exponent energy of different graph classes.

Keywords: Degree of a vertex, Degree exponent matrix, Degree exponent energy, Line graph.

Mathematics Subject Classification: 05C50, 05C12

1. Introduction

Throughout this paper G will denote a simple (no loops or multiple edges), undirected graph with n-vertices and m-edges. Let {v1, v2, . . . , vn} be the set of vertices of G and if two vertices v_i and v_j of G are adjacent, then we write v_i ~ v_j. For v_i ∈ V(G), the degree of the vertex v_i, denoted by di, is the number of vertices adjacent to v_i.
Motivated by the work on adjacency polynomial, Laplacian polynomial, distance polynomial extensively studied by the researchers, Ramane et al. defined Degree exponent matrix as \[ DE(G) = [de_{ij}], \]
in which \[ de_{ij} = \begin{cases} 
    d_i^j, & \text{if } i \neq j \\
    0, & \text{if } i = j.
\end{cases} \]

To study the properties of graphs, we introduced here another matrix of a graph called degree exponent matrix with respect to adjacency matrix which is defined as

\[
DE_A(G) = (de_{ij})_{n \times n},
\]
in which

\[
de_{ij} = \begin{cases} 
    (d_i)^{v_j} & \text{if } v_i \sim v_j \\
    0 & \text{otherwise}.
\end{cases}
\]

Denote the eigenvalues of the \( DE \) matrix of \( G \) by \( \lambda_1, \lambda_2, \ldots, \lambda_n \) and label them in nonincreasing order. Similar to the characteristic polynomial of matrix. We consider the \( DE \) characteristic polynomial of \( G \) as \( \det (\lambda I - DE_A(G)) \) which is equal to \( \prod_{i=1}^{n} (\lambda - \lambda_i) \). The \( DEE \) is defined as \( DEE_A(G) = \sum_{i=1}^{n} |\lambda_i| \).

The adjacency matrix of \( G, A(G) = [a_{ij}] \) is an \( n \times n \) matrix, where \( a_{ij} = 1 \) if \( v_i \sim v_j \) and 0 otherwise. Thus, \( A \) is a symmetric \((0, 1)\)-matrix with a real eigenvalues and zeros on the diagonal. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( A \), then we have \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) and \( \lambda_1 + \lambda_2 + \ldots + \lambda_n = 0 \). The energy of \( G \) was first defined by Gutman in 1978 as the sum of all absolute values of the eigenvalues of \( A(G) \) \[ E(G) = \sum_{i=1}^{n} |\lambda_i| \] .

The concept of energy originated in chemistry. Huckel molecular orbital (HMO) theory is a field of theoretical chemistry where graph eigenvalues occurred. The carbon atoms of a hydrocarbon system correspond to vertices of a graph associated with the molecule. From Huckel theory, the energy of a molecular graph is equal to the total \( \pi \)-electron energy of a conjugated hydrocarbon [10]. For details on the energy of a graph \( G \), one can refer the book [15] and the references cited there in. There are many other kinds of graph energies, such as incidence energy [2, 3], distance energy [24], Laplacian energy [8], matching energy [6, 12 and 13], Randic energy [21], skew energy [19] and Reciprocal complementary distance energy [20].

2. DEA-Polynomial and DEA-Energy of Specific Graphs

**Theorem 2.1.** Let \( G \) be an \( r \)-regular graph on \( n \)-vertices. Then the \( DE \) characteristic polynomial of \( G \) is
Proof. If $G$ is an $r$-regular graph on $n$-vertices, then by the definition of degree exponent matrix of $G$

$$DE_A(G, \lambda) = r^n \varphi(G, \lambda/r^r).$$

Therefore,

$$DE_A(G, \lambda) = \det(\lambda I - r^r A(G)) = (r^r)^n \varphi(G, \lambda/r^r).$$

Corollary 2.2. The DE-polynomial of cycle graph is

$$DE_A(C_n, \lambda) = r^n \prod_{j=0}^{n-1} \left[ \frac{\lambda}{r^r} - 2\cos\left(\frac{2\pi j}{n}\right) \right]$$

Theorem 2.3. [18]

For $n \geq 2$,

(i) The DE polynomial of the complete graph $K_n$ with $n$ vertices is

$$DE_A(K_n, \lambda) = (\lambda + (n-1)^{(n-1)})^{n-1} \left[ \lambda - (n-1)^{(n-1)}(n-1) \right]$$

(ii) The DEE of $K_n$ is

$$DEE_A(K_n) = 2(n-1)^n$$

Theorem 2.4. [20] For $n \geq 3$, the general Randić characteristic polynomial of the cycle graph $C_n$ satisfy

$$GR(C_n, \lambda) = \lambda \Lambda_{n-1} - 2(16)^{\alpha} \Lambda_{n-2} - (4)^{n+2}, \quad \alpha \in R,$$

where for every $k \geq 3$, $\Lambda_k = \lambda \Lambda_{k-1} - (16)^{\alpha} \Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - (16)^{\alpha}$.

Corollary 2.5. For $n \geq 3$, the DE characteristic polynomial of the cycle graph $C_n$ satisfy

$$DE_A(C_n, \lambda) = \lambda \Lambda_{n-1} - 32 \Lambda_{n-2} - 2^{2n+1},$$

where for every $k \geq 3$, $\Lambda_k = \lambda \Lambda_{k-1} - 16 \Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - 16$. 
Proof. Substituting \( \alpha = 1 \) in Theorem 1.2, the result follows.

**Theorem 2.6.** For \( n \geq 5 \), the DE characteristic polynomial of the path graph satisfy

\[
DE_A(P_n, \lambda) = \lambda^2 \Lambda_{n-2} - 4\lambda \Lambda_{n-3} + 4\Lambda_{n-4},
\]

where for every \( k \geq 3 \), \( \Lambda_k = \lambda \Lambda_{k-1} - 16\Lambda_{k-2} \) with \( \Lambda_1 = \lambda \) and \( \Lambda_2 = \lambda^2 - 16 \).

**Proof.** For every \( k \geq 3 \), consider

\[
B_k = \begin{bmatrix}
\lambda & -2^2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-2^2 & \lambda & -2^2 & 0 & \ldots & 0 & 0 & 0 \\
0 & -2^2 & \lambda & -2^2 & \ldots & 0 & 0 & 0 \\
0 & 0 & -2^2 & \lambda & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \lambda & -2^2 & 0 \\
\end{bmatrix}_{k \times k},
\]

and let \( \Lambda_k = \det(B_k) \). It is easy to see that \( \Lambda_k = \lambda \Lambda_{k-1} - 16\Lambda_{k-2} \).

Suppose that \( DE_A(P_n : \lambda) = \det(\lambda I - DE_A(P_n)) \). We have

\[
DE_A(P_n, \lambda) = \begin{bmatrix}
\lambda & -1^2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1^2 & \lambda & -2^2 & 0 & \ldots & 0 & 0 & 0 \\
0 & -2^2 & \lambda & -2^2 & \ldots & 0 & 0 & 0 \\
0 & 0 & -2^2 & \lambda & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \lambda & -2^2 & 0 \\
\end{bmatrix}_{n \times n}
\]
Lemma 2.7. [7] If $M$ is a nonsingular square matrix, then

$$\det \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det(M) \det(Q - P M^{-1} N).$$

Theorem 2.8. For any positive integers $p, q$,

(i) The DE polynomial of complete bipartite graph $K_{p,q}$ is

$$DE_A(K_{p,q}, \lambda) = \lambda^{p+q-2}(\lambda^2 - m^{p+1} n^{q+1}).$$

(ii) $D E E_A(K_{p,q}) = 2\sqrt{m^{p+1} n^{q+1}}$.

Proof. The DE polynomial of $K_{p,q}$ is

$$|\lambda I - D E_A(K_{p,q})| = \begin{vmatrix} \lambda I_p & -n^m J_{p\times q} \\ -m^n J_{q\times p} & \lambda I_q \end{vmatrix}. $$

Using Lemma 2.3 we have

$$= |\lambda I_p| \left| \lambda I_q - p^n J_{q\times p} \frac{1}{\lambda} q^p J_{p\times q} \right| $$

$$= \lambda^{p-q} \left| \lambda^2 - p^n q^p \lambda J_q \right| \quad \text{since} \quad J_{q\times p} J_{p\times q} = p J_q.$$
Since the eigenvalues of $J_n$ are $n$ (once) and $0$ ($n - 1$ times), the eigenvalues of $p^{q+1} q^{p} \ J_q$ are $p^{q+1}$ ($once$) and $0$ ($q - 1$ times). Therefore

$$DE_A(K_{p,q}, \lambda) = \lambda^{p+q-2}(\lambda^2 - p^{q+1}q^{p+1}).$$

(ii) It follows from Part (i).

**Corollary 2.9.** For $n \geq 2$,

(i) The DE polynomial of the star graph $S_n = K_{1, n-1}$ is

$$DE_A(S_n, \lambda) = \lambda^{n-2}((\lambda^2 - (n - 1)^2)).$$

(ii) The DEE of $S_n$ is

$$DEE_A(S_n) = 2(n - 1).$$

Let $n$ be any positive integer and $F_n$ be friendship graph with $2n + 1$ vertices and $3n$ edges. In other words, the friendship graph $F_n$ is a graph that can be constructed by coalescence $n$ copies of the cycle graph $C_3$ of length 3 with common vertex. The Friendship theorem of Erdős et al. [15], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. The Fig. 1 shows some examples of friendship graphs. Here we shall compute the DEE of friendship graphs.

![Friendship Graphs](image)

**Fig. 1.** Friendship graphs $F_2$, $F_3$ and $F_n$ respectively

**Theorem 2.10.** For $n \geq 2$,

(i) The DE polynomial of friendship graph $F_n$ is

$$DE_A(F_n, \lambda) = (\lambda^2 - 16)^{n-1}(\lambda + 4)(\lambda^2 - 4\lambda - 8n^3)^n$$

(ii) The DE energy of friendship graph $F_n$ is
\[ DEE_A(F_n) = \begin{cases} 8n & \text{if } n^3 4^n \leq 0 \\ 8(n - 1) + 4 + 2\sqrt{1 + 2n^3 4^n} & \text{if } n^3 4^n > 0 \end{cases} \]

**Proof.** The DE matrix of \( F_n \) is

\[
DE_A(F_n) = \begin{bmatrix}
0 & (2n)^2 & (2n)^2 & \cdots & (2n)^2 & (2n)^2 \\
2^{2n} & 0 & 2^2 & \cdots & 0 & 0 \\
2^{2n} & 2^2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2^{2n} & 0 & 0 & \cdots & 0 & 2^2 \\
2^{2n} & 0 & 0 & \cdots & 2^2 & 0
\end{bmatrix}_{(2n+1) \times (2n+1)}
\]

Now, for computing \([\lambda I - DE_A(F_n)]\), we consider its first row. The cofactor of the first array in this row is

\[
\begin{bmatrix}
\lambda & -2^2 & \cdots & 0 & 0 \\
-2^2 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda & -2^2 \\
0 & 0 & \cdots & -2^2 & \lambda
\end{bmatrix}_{(2n) \times (2n)}
\]

and the cofactor of another arrays in the first row are similar to

\[
\begin{bmatrix}
-2^{2n} & -2^2 & \cdots & 0 & 0 \\
-2^{2n} & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-2^{2n} & 0 & \cdots & \lambda & -2^2 \\
-2^{2n} & 0 & \cdots & -2^2 & \lambda
\end{bmatrix}_{(2n) \times (2n)}
\]

Now solving the above two matrix, we get

\[
DE(F_n, \lambda) = \lambda(\lambda^2 - 16)^n + 8n^3 \left[ -4^n(\lambda + 4)(\lambda^2 - 16)^{(n-1)} \right]
\]

(iii) It follows from Part (i)
Let \( n \) be any positive integer and \( D_4^n \) be Dutch Windmill graph with \( 3n + 1 \) vertices and \( 4n \) edges. In other words, the graph \( D_4^n \) is a graph that can be constructed by coalescence \( n \) copies of the cycle graph \( C_4 \) of length 4 with a common vertex. The Fig. 2 shows some examples of Dutch Windmill graphs. Here we shall compute the DEE of Dutch Windmill graphs.

**Theorem 2.11.** For \( n \geq 2 \),

(i) The DE polynomial of friendship graph \( D_4^n \) is

\[
DE_A(D_4^n, \lambda) = \lambda^{n+1}(\lambda^2 - 32)^{n-1}[\lambda^2 - (32 + 8n^34^n)].
\]

(ii) The \( \text{DEE}_A(D_4^n) = 2\sqrt{32}(n - 1) + 2\sqrt{32} + 8n^34^n \).

**Proof.** The DE matrix of \( D_4^n \) is

\[
DE_A(D_4^n) = \begin{bmatrix}
0 & (2n)^2 & (2n)^2 & 0 & \ldots & (2n)^2 & (2n)^2 & 0 \\
2^{2n} & 0 & 0 & 2^2 & \ldots & 0 & 0 & 0 \\
2^{2n} & 0 & 0 & 2^2 & \ldots & 0 & 0 & 0 \\
0 & 4^n & 2^2 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2^{2n} & 0 & 0 & 0 & \ldots & 0 & 0 & 2^2 \\
2^{2n} & 0 & 0 & 0 & \ldots & 0 & 0 & 2^2 \\
0 & 0 & 0 & 0 & \ldots & 2^2 & 2^2 & 0 \\
\end{bmatrix}_{(3n+1) \times (3n+1)}.
\]

Let \( A = \begin{bmatrix}
\lambda & 0 & -4 \\
0 & \lambda & -4 \\
-4 & -4 & \lambda \\
\end{bmatrix} \), \( B = \begin{bmatrix}
-2^{2n} & 0 & 0 \\
-2^{2n} & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \) and \( C = \begin{bmatrix}
-2^{2n} & 0 & -2^2 \\
-2^{2n} & \lambda & -2^2 \\
0 & -2^2 & \lambda \\
\end{bmatrix} \).

Then
Now, by the straightforward computation we have the result.

(ii) It follows from Part(i).

Let \( n \) be any positive integer and \( K_4^n \) be Dutch Windmill graph with \( 4n + 1 \) vertices and \( 6n \) edges. In otherwords, the graph \( K_4^n \) is a graph that can be constructed by coalescence \( n \) copies of the complete graph \( K_4 \) with a common vertex. The Fig. 3 shows some examples of \( K_4^n \)– Windmill graphs.

![Fig. 3. K4− Windmill graph](image)

**Theorem 2.12.** For \( n \geq 2 \),

(i) The DE polynomial of \( K_4^n \)– Windmill graph is

\[
DE_A(K_4^n, \lambda) = (\lambda + 3^3)^{2n} (\lambda - 2(3)^3) \left( \lambda - \left[ 9(3 + \sqrt{9 + n^4 3^{3n}}) \right] \right)
\]

(ii) The DE energy of \( K_4^n \)– Windmill graph is

\[
DEE_A(K_4^n) = \begin{cases} 
108n & \text{if } n^4 3^3 n \leq 0 \\
54(2n - 1) + 18\sqrt{9 + n^4 3^{3n}} & \text{if } n^4 4^n > 0 
\end{cases}
\]
Proof

\[ DE_A(K^n_4) = \begin{bmatrix}
0 & (3n)^3 & (3n)^3 & \ldots & (3n)^3 \\
(3n)^3 & 0 & 3^3 & \ldots & 3^3 \\
(3n)^3 & 3^3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(3n)^3 & 0 & 3^3 & \ldots & 3^3 \\
(3n)^3 & 0 & 3^3 & \ldots & 3^3 \\
\end{bmatrix} \]

Let \( A = \begin{pmatrix}
\lambda & -3^3 & -3^3 \\
-3^3 & \lambda & -3^3 \\
-3^3 & -3^3 & \lambda \\
\end{pmatrix} \), \( B = \begin{pmatrix}
-3^{2n} & 0 & 0 \\
-3^{2n} & 0 & 0 \\
-3^{2n} & 0 & 0 \\
\end{pmatrix} \) and \( C = \begin{pmatrix}
-3^{2n} & -3^3 & -3^3 \\
-3^{2n} & \lambda & -3^3 \\
-3^{2n} & -3^3 & \lambda \\
\end{pmatrix} \).

Then

\[
det(\lambda I - DE_A(K^n_4)) = \lambda (det(A))^n + 3n(3n)^3 det
\]

Now, by the straightforward computation we have the result.

\( (ii) \) It follows from Part(i).

\[
\text{Fig. 4. Double star } S_{3,3}
\]

For \( p, q \geq 1 \) the double star \( S(p, q) \) is the graph on the points \( \{v_0, v_i, \ldots, v_p, w_0, w_i, \ldots, w_q\} \) with lines

\[ \{(v_0, w_0), (v_0, v_i), (w_0, w_j) : 1 \leq i \leq p, 1 \leq j \leq q\} \].

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Theorem 2.13. For \( p, q \geq 1 \)

(i) The DE-polynomial of double star graph \( S(p, q) \) is

\[
DE_A(S(p, q), \lambda) = \lambda^{p+q-4} \left[ \lambda^4 - \lambda^2 (p(p-1) + q(q-1) + p^q q^p) + (pq)(p-1)(q-1) \right].
\]

(ii) The \( DE_A(S(p, q)) = \sqrt{2} \sqrt{X + \sqrt{X^2 - 4pq(p-1)(q-1)}} + \sqrt{2} \sqrt{X - \sqrt{X^2 - 4pq(p-1)(q-1)}}, \]

where \( X = p(p-1) + q(q-1) + p^q q^p. \)

Proof. The \( DE- \) matrix of \( S(p, q) \) is

\[
DE_A(S(p, q)) = \begin{bmatrix}
\lambda & -p^q & -p & \ldots & -p & 0 & \ldots & 0 \\
-p & \lambda & 0 & \ldots & 0 & -q & \ldots & -q \\
-q^p & 0 & \lambda & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1^p & 0 & 0 & \ldots & \lambda & 0 & \ldots & 0 \\
0 & -1^q & 0 & \ldots & 0 & \lambda & \ldots & 0 \\
0 & -1^q & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \lambda \\
\end{bmatrix}_{(p+q) \times (p+q)}.
\]

Using Lemma 2.3,

\[
= \lambda^{p+q-4} \begin{bmatrix}
\lambda^2 - p(p-1) & -\lambda q^p \\
-\lambda p^q & \lambda^2 - q(q-1) \\
\end{bmatrix}
\]

Now, by the straightforward computation we have the result.

(ii) It follows from Part (i).

3. Degree Exponent Eigenvalues of Line Graph of Regular Graphs

The line graph of \( G \) denoted by \( L(G) \) is a graph whose vertices corresponds to the edges of \( G \) and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges are adjacent in \( G \) [11]. The \( k \)-th line graph of \( G \) is defined as \( L^k(G) = L(L^{k-1}(G)) \) where \( L^0(G) \equiv G \) and \( L^1(G) = L(G) \). The line graph of a regular graph is a regular graph. If \( G \) is a regular graph of order \( n_0 \) and of degree \( r_0 \), then \( L(G) \)
is a regular graph of order $n_1 = \frac{1}{2} n_0 r_0$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [4, 5]:

$$n_k = \frac{1}{2} n_{k-1} r_{k-1} \text{ and } r_k = 2r_{k-1} - 2$$

where $n_i$ and $r_i$ stand for the order and degree of $L^i(G)$, $i = 0, 1, 2, \ldots$. Therefore [4, 5],

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \quad (1)$$

and

$$r_k = 2^k r_0 - 2^{k+1} + 2. \quad (2)$$

**Theorem 3.1.** [23] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $L(G)$ are

$$\lambda_i + r - 2 \quad i = 1, 2, \ldots, n, \text{ and } -2 \quad n(r - 2)/2 \text{ times.}$$

**Theorem 3.2.** [22] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r$, then the adjacency eigenvalues of $\bar{G}$, the complement of $G$ are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.

**Theorem 3.3.** If $G$ is a regular graph of order $n$ and of degree $r$, then the degree exponent eigenvalues of $L(G)$ are

$$(2r - 2)^{(2r-2)}(\lambda_i + r - 2) \quad i = 1, 2, \ldots, n, \quad \text{and}$$

$$-2(2r - 2)^{(2r-2)} \quad n(r - 2)/2 \text{ times.}$$

**Proof.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r \geq 4$. Then by Theorem 3.1 the adjacency eigenvalues of $L(G)$ are

$$\lambda_i + r - 2 \quad i = 1, 2, \ldots, n \quad \text{and}$$

$$-2 \quad n(r - 2)/2 \text{ times} \quad (3)$$
Since $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$, from Eq. 3 and by Theorem 2.1 the degree exponent eigenvalues of $L(G)$ are

$$
(2r - 2)^{(2r-2)} (\lambda_i + r - 2) \quad i = 1, 2, \ldots, n, \quad \text{and} \\
-2(2r - 2)^{(2r-2)} n(r - 2)/2 \text{ times.}
$$

**Theorem 3.4.** If $G$ is a regular graph of order $n$ and of degree $r$, then the degree exponent eigenvalues of $L^2(G)$ are

$$
(4r - 6)^{(4r-6)} (\lambda_i + 3r - 6) \quad i = 1, 2, \ldots, n \quad \text{and} \\
(4r - 6)^{(4r-6)} (2r - 6) n(r - 2)/2 \text{ times and} \\
-2 (4r - 6)^{(4r-6)} nr(r - 2)/2 \text{ times.}
$$

**Proof.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the adjacency eigenvalues of a regular graph $G$ of order $n$ and of degree $r \geq 4$. Then by Theorem 3.1 the adjacency eigenvalues of $L(G)$ are

$$
\lambda_i + r - 2 \quad i = 1, 2, \ldots, n \quad \text{and} \\
-2 \quad n(r - 2)/2 \text{ times}
$$

Since $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$, from Eq. 4 the adjacency eigenvalues of $L^2(G)$ are

$$
\lambda_i + 3r - 6 \quad i = 1, 2, \ldots, n \quad \text{and} \\
2r - 6 \quad n(r - 2)/2 \text{ times} \quad \text{and} \\
-2 \quad nr(r - 2)/2 \text{ times}
$$

Since $L^2(G)$ is a regular graph of order $nr(r - 1)/2$ and of degree $4r - 2$, from Theorem 2.1 and Eq. (4) the degree exponent eigenvalues of $L^2(G)$ are

$$
(4r - 6)^{(4r-6)} (\lambda_i + 3r - 6) \quad i = 1, 2, \ldots, n \quad \text{and} \\
(4r - 6)^{(4r-6)} (2r - 6) n(r - 2)/2 \text{ times and} \\
-2 (4r - 6)^{(4r-6)} nr(r - 2)/2 \text{ times.}
$$

**Theorem 3.5.** If $G$ is a regular graph of order $n$ and of degree $r$, then the degree exponent eigenvalues of $\bar{G}$ are
\[(n - r - 1)^{(n-r-1)}(-\lambda_i - 1) \quad i = 2, \ldots, n, \quad \text{and} \]
\[(n - r - 1)^{(n-r-1)}(n - r - 1) \quad n(r-2)/2 \quad \text{times}. \]

**Proof.** By the Theorem 2.1 and Theorem 3.2 the result follows.

**Theorem 3.6.** If \(G\) is a regular graph of order \(n\) and of degree \(r \geq 3\), then the degree exponent eigenvalues of \(\overline{L(G)}\) are

\[
\begin{align*}
\left(\frac{nr}{2} - 2r + 1\right)^{(nr/2-2r+1)} (-\lambda_i - r + 1) & \quad i = 2, \ldots, n \quad \text{and} \\
\left(\frac{nr}{2} - 2r + 1\right)^{(nr/2-2r+1)} n(r-2)/2 \quad \text{times} & \quad \text{and} \\
\left(\frac{nr}{2} - 2r + 1\right)^{(nr/2-2r+1)} \left(\frac{nr}{2} - 2r + 1\right) n(r-2)/2 \quad \text{times}. & \\
\end{align*}
\]

**Proof.** By the Theorem 3.1, 3.2 and Theorem 2.1 the result follows.

**Theorem 3.7.** If \(G\) is a regular graph of order \(n\) and of degree \(r \geq 3\), then the degree exponent eigenvalues of \(\overline{L^2(G)}\) are

\[
\begin{align*}
\left[\frac{nr(r-1)}{2} - (4r - 5)\right] - \lambda_i - 3r + 5 & \quad i = 2, 3, \ldots, n \quad \text{and} \\
\left[\frac{nr(r-1)}{2} - (4r - 5)\right] - 2r + 5 & \quad n(r-2)/2 \quad \text{times} \quad \text{and} \\
\left[\frac{nr(r-1)}{2} - (4r - 5)\right] & \quad nr(r-2)/2 \quad \text{times} \quad \text{and} \\
\left[\frac{nr(r-1)}{2} - (4r - 5)\right]^2 & \\
\end{align*}
\]

**Proof.** Let \(\lambda_1, \lambda_2, \ldots, \lambda_n\) be the adjacency eigenvalues of a regular graph \(G\) of order \(n\) and of degree \(r \geq 3\). Then the adjacency eigenvalues of \(L^2(G)\) are as given in Eq. (4).

Since \(L^2(G)\) is a regular graph of order \(nr(r - 1)/2\) and of degree \(4r - 6\), from Theorem 3.2 and Eq. (4) the adjacency eigenvalues of \(\overline{L^2(G)}\) are

\[
\begin{align*}
-\lambda_i - 3r + 5 & \quad i = 2, 3, \ldots, n \quad \text{and} \\
-2r + 5 & \quad n(r-2)/2 \quad \text{times} \quad \text{and} \\
1 & \quad nr(r-2)/2 \quad \text{times} \quad \text{and} \\
(nr(r-1)/2 - 4r + 5 & \\
\end{align*}
\]

Since \(\overline{L^2(G)}\) is a regular graph of order \(nr(r - 1)/2\) and of degree \((nr(r - 1)/2) - 4r + 5\), from Theorem 2.1 and Eq. 5 the degree exponent eigenvalues of \(L^2(G)\) are
From the section 3 we can easily find the degree exponent energy of line graph of regular graph.

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