The Banach Numerical Range for Finite Linear Operators

Priscah M. Ohuru¹, Sammy W. Musundi²*

¹Faculty of Pure and Applied Sciences, Kisii University, P.O. Box 408-40200, Kisii
²Faculty of Science, Engineering & Technology, Chuka University, P.O. Box 109-60400, Kenya

*Author to whom correspondence should be addressed; E-Mail: sammusundi@yahoo.com

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Abstract: The numerical range has been a subject of interest to many researchers and scholars in the recent past. Based on the research outputs, many results have been obtained. Besides, several generalizations of the classical numerical range have also been made. The recent developments have focused on the theory of operators on Hilbert spaces. The determination of the numerical ranges of linear and nonlinear operators have been given in both the Hilbert and Banach spaces. In addition, results of these numerical ranges have been extended to the case of two operators in both spaces. It is important to note that more generalizations have been made in Hilbert spaces as compared to those that have been made in the Banach spaces. The Banach space has two major numerical ranges which are: the spatial and algebraic numerical ranges. This research focuses on determining the numerical range for a finite number of linear operators in the Banach space based on the classical definition. Properties which hold for the classical numerical range have been shown to hold for the Banach space numerical range. The property of convexity has been established using the Toeplitz-Hausdorff theorem under the condition that the Banach space is smooth. Furthermore, the numerical radius and the spectrum of these operators have also been determined.

Keywords: Classical Numerical Range; The Banach Space Numerical Range, Radius, and Spectrum.

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1. Introduction

The concept of numerical range also known as the field of values was first introduced by Toeplitz in 1918 acting on a finite dimensional Hilbert space and he defined the Numerical range of an operator T as the set

\[ W(T) = \{ <Tx, x > : x \in H, \|x\| = 1 \} \]

One of the most important properties of the classical numerical range is convexity which was proved by Haursdoff in 1919 (Toeplitz-Haursdoff Theorem). Consequently, whenever there is a new generalization of the numerical range, it is of interest to know if it is convex. Despite giving a deeper understanding of the classical result, the study of convexity for generalized numerical ranges is useful in further development of theory.

Besides convexity being a celebrated result, establishment of the relationship between the numerical range and spectrum has also been of great interest to many researchers in this area. Lumer [3], defined the numerical range of a linear operator (linear transformation) T as the set of numbers \( W(T) = \{ <Tx, x >: < x, x > = 1 \} \) in a Semi-inner product space, which extended the classical numerical range and further showed that the spectrum of the operator T is contained in the closure of this numerical range.

Williams [6], presented the spectra of the product of two bounded linear operators on a Banach space and certain nonlinear transformations on a real or complex Hilbert space. Zarantonello [7], introduced the numerical range for a nonlinear operator and proved that the closure of the Numerical range contains the spectrum in the theory of linear operators in Hilbert space also holds for nonlinear operators.

Amelin [1], introduced the concept of the numerical range for two linear operators in a Hilbert space and obtained results for index stability of a Fredholm operator. Mecheri [4], defined the numerical range of a linear operator as a subset of the complex plane whose geometric properties defines the operator and proceeded to show that every paranormal operator is normaloid. Sahu et al.[5], studied the numerical range for two operators both linear and nonlinear in semi-inner product spaces. The numerical range for one and two linear operators in Hilbert and Banach spaces has been covered. This research is devoted to extend the previous results on the numerical range for two linear operators to a finite number of linear operators in the Banach space.

2. Main Results

2.1. The Numerical Range for Finite Linear Operators in the Banach Space
In this section, we generalize the results for the numerical range of a linear operator defined by Lumer [3], as the set \( V(T) = \{ x^*(Tx) : x^* \in X^*, x \in X, \ x^*(x) = 1 \} \) to a finite case. We first give the definition of the numerical for finite linear operators which is very important in obtaining our set results.

**Definition 2.1.1:**

Let \( X \) be a Banach space and \( T_i \) for \( i = 1, 2, 3, \ldots, n \), be linear operators. The numerical range for \( T_i \) is the set
\[
V(T_i) = \left\{ \sum_{i=1}^{n} x^*(T_i x) : x^* \in X^*, x \in X, \ x^*(x) = 1 \right\},
\]
with \( X^* \) the dual space of \( X \).

**Remark 2.1.2:**

Given \( i = 1 \), then defined numerical range is deduced to
\[
V(T_1) = \{ x^*(T_1 x) : x^* \in X^*, x \in X, \ x^*(x) = 1 \},
\]
giving the definition of the numerical range for one operator.

**Remark 2.1.3:**

By the Representation Theorem on the dual of a Hilbert space, the definition of the numerical range for a bounded linear operator in the Banach space coincides with that of the classical numerical range in the Hilbert space.

In the following proposition, we establish important properties of the defined numerical range.

**Proposition 2.1.4:**

Let \( X \) be a Banach Space, \( T, S \) be bounded linear operators and \( U \) be an isometry, then

i) \( V(T_i) \) is non-empty

ii) \( V(T_i) \) is unitary invariant, that is, \( V(U_i^{-1}T_i U_i) = V(T_i) \)

iii) \( V(\alpha_i I + T_i) = \alpha_i + V(T_i) \)

iv) \( V(\alpha_i T_i + \beta_i S_i) \subseteq \alpha_i V(T_i) + \beta_i V(S_i) \)

v) \( V(T_i) \subseteq V(T_i^*) \subseteq \overline{V(T_i)} \)

vi) \( V(T_i) \) is a convex set

**Proof**

i) Since \( T_i \), for \( i = 1, 2, 3, \ldots, n \) are linear operators and \( X \neq 0 \), let \( x \in X \),
\[
x^* \in X^* \text{ such that } x^*(x) = 1,
\]
then
\[
V(T_i) = \left\{ \sum_{i=1}^{n} x^*(T_i x) : x^* \in X^*, x \in X, \ x^*(x) = 1 \right\} \neq \emptyset,
\]
that is, we must have an element \( x \in X \) and therefore our set \( V(T_i) \) cannot be empty.

ii) Let \( \lambda \in V(U_i^{-1}T_i U_i) \), then \( \lambda = x^*(U_i^{-1}T_i U_i)x \), \( T_i \) are linear operators and \( U_i \) isometries in the Banach space, hence we have,
\[
\lambda = x^*(U_i^{-1}T_i U_i x) = x^*(T_i U_i^{-1} U_i x) = x^*(T_i x U_i^{-1} U_i x)
\]

\[
= x^*(T_i x). \ x^*(U_i^{-1} U_i x), \text{ but } U_i^{-1} U_i = I
\]

\[
= x^*(T_i x). \ x^*(I x) = x^*(T_i x). \ I
\]

\[
\lambda = x^*(T_i x)
\]

\[
\Rightarrow \lambda \in V(T_i)
\]

\[
\text{Conversely,}
\]

\[
\text{Let } \lambda \in V(T_i)
\]

\[
\Rightarrow \lambda = x^*(T_i x) = x^*(T_i x). \ I = x^*(T_i x). \ x^*(I x)
\]

\[
= x^*(T_i x). \ x^*(U_i^{-1} U_i x), \text{ for } U_i^{-1} U_i = I
\]

\[
= x^*(T_i x U_i^{-1} U_i x) = x^*(T_i U_i^{-1} U_i x)
\]

\[
\lambda = x^*(U_i^{-1} T_i U_i x)
\]

\[
\Rightarrow \lambda \in V(U_i^{-1} T_i U_i)
\]

\[
\text{Hence, (i) and (ii) } \Rightarrow V(U_i^{-1} T_i U_i) = V(T_i).
\]

\[
\text{iii) } V(\alpha I + T_i) = \{ \sum_{i=1}^{n} x^*(\alpha I + T_i) x : x^* \in X^*, x \in X, x^*(x) = 1 \}
\]

\[
= \{ \sum_{i=1}^{n} x^*(\alpha I + T_i) x : x^* \in X^*, x \in X, x^*(x) = 1 \}
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\[
= \{ \sum_{i=1}^{n} x^*(\alpha I + T_i) x : x^* \in X^*, x \in X, x^*(x) = 1 \}
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\[
= \{ \sum_{i=1}^{n} x^*(\alpha I + T_i) x : x^* \in X^*, x \in X, x^*(x) = 1 \}
\]

\[
= \{ \sum_{i=1}^{n} \alpha I + x^*(T_i x) : x^* \in X^*, x \in X, x^*(x) = 1 \}
\]

\[
= \alpha I + V(T_i)
\]

\[
\text{iv) } \lambda \in V(\alpha_i T_i + \beta_i S_i)
\]

\[
\lambda = x^*(\alpha_i T_i + \beta_i S_i) x = x^*(\alpha_i T_i x + \beta_i S_i x) = x^*(\alpha_i T_i x) + x^*(\beta_i S_i x)
\]

\[
= \alpha_i x^*(T_i x) + \beta_i x^*(S_i x)
\]

\[
\Rightarrow \lambda \in \alpha_i V(T_i) + \beta_i V(S_i)
\]

\[
\Rightarrow V(\alpha_i T_i + \beta_i S_i) \subseteq \alpha_i V(T_i) + \beta_i V(S_i).
\]

\[
\text{v) Let } \lambda \in V(T_i) \text{ and } T_i \text{ be unitary operators with } T_i^* T_i = I
\]

\[
\lambda = x^*(T_i x) = x^*(T_i I x), \text{ for } I \text{ an identity operator in } X
\]

\[
= x^*(T_i x). \ x^*(I x) = x^*(T_i^* T_i x) = x^*(T_i^* T_i I x)
\]

\[
= x^*(I x). \ x^*(T_i^* I x) = x^*(x). \ x^*(T_i^* x) = 1 \ x^*(T_i^* x) = x^*(T_i^* x)
\]

\[
\Rightarrow \lambda \in V(T_i^*)
\]

\[
\text{Hence } V(T_i) \subseteq V(T_i^*).
\]

Since \( V(T_i) \subseteq \overline{V(T_i)} \) always holds, then \( V(T_i) \subseteq V(T_i^*) \subseteq \overline{V(T_i)} \) is true.

\[
\text{vi) If we consider the set } V(T) = \{ x^*(T x) : x^* \in X^*, x \in X, x^*(x) = 1 \}, \text{ it has been shown that this set is convex under the condition that the Banach space } X \text{ is smooth. A space } X \text{ is said to be}
\]
smooth if for every $x \in X, x * \in X * \exists a$ unique norm-one functional $\varphi$ such that $\| \varphi(x) \| = \| x \|, x * (x) = 1$ (Chidume, [2]).

Similarly, we shall show that the set

$$V(T_i) = \left\{ \sum_{i=1}^{n} x^*(T_i x) : x^* \in X^*, x \in X, \quad x^*(x) = 1 \right\}$$

is convex under the same condition.

Let $\lambda_1, \lambda_2 \in V(T_i) \forall i = 1,2,...,n$.

We aim to show that

$$\alpha \lambda_1 + (1 - \alpha) \lambda_2 \in V(T_i), 0 < \alpha \leq 1.$$ 

Since $X$ is a smooth space, then there exists a unique map $\varphi$ from $X$ to $X^*$ such that $\| \varphi(x) \| = \| x \|$.

Let $f_1, f_2 \in X^*$ with $f_1(\sum_{i=1}^{n} x^*(T_i x)) = \lambda_1, f_2(\sum_{i=1}^{n} x^*(T_i x)) = \lambda_2$ and $f_1(I) = 1 = f_1 \parallel, f_2(I) = 1 = f_2 \parallel$.

Define $\varphi$ on $X^*$ by

$$\varphi(\sum_{i=1}^{n} T_i) = \alpha f_1(\sum_{i=1}^{n} x^*(T_i x)) + (1 - \alpha) f_2(\sum_{i=1}^{n} x^*(T_i x)).$$

We can show that $\varphi$ is linear.

Let $\beta_1, \beta_2 \in \mathbb{K}$, then

$$\varphi(\beta_1 \sum_{i=1}^{n} T_i + \beta_2 \sum_{j=1}^{m} T_j) = \alpha f_1\left( \beta_1 \sum_{i=1}^{n} x^*(T_i x) + \beta_2 \sum_{j=1}^{m} x^*(T_j x) \right) + (1 - \alpha) f_2 \left( \beta_1 \sum_{i=1}^{n} x^*(T_i x) + \beta_2 \sum_{j=1}^{m} x^*(T_j x) \right)$$

$$= \alpha \beta_1 f_1(\sum_{i=1}^{n} x^*(T_i x)) + \alpha \beta_2 f_1(\sum_{j=1}^{m} x^*(T_j x)) + (1 - \alpha) \beta_1 f_2(\sum_{i=1}^{n} x^*(T_i x)) + (1 - \alpha) \beta_2 f_2(\sum_{j=1}^{m} x^*(T_j x))$$

$$= \beta_1 \varphi(\sum_{i=1}^{n} T_i) + \beta_2 \varphi(\sum_{j=1}^{m} T_j)$$

Hence, $\varphi$ is linear.

Next we show that $\| \varphi \| = 1$.

Now, $\varphi(I) = \alpha f_1(I) + (1 - \alpha) f_2(I) = 1$.
1 = ∥ φ(I) ∥ ≤ ∥ φ ∥ ∥ I ∥

1 ≤ ∥ φ ∥ .................................................................(i)

Conversely,

\[ |φ(∑_{i=1}^{n} T_i)| = |αf_1 \{∑_{i=1}^{n} x^*(T_i x)\} + (1 - α)f_2 \{∑_{i=1}^{n} x^*(T_i x)\}| \]

\[ \leq |αf_1 \left(∑_{i=1}^{n} x^*(T_i x)\right)| + |(1 - α)f_2 \left(∑_{i=1}^{n} x^*(T_i x)\right)| \]

\[ \leq |α| ∥ f_1 ∥ ∥ \left(∑_{i=1}^{n} x^*(T_i x)\right)∥ + |(1 - α)| ∥ f_2 ∥ ∥ \left(∑_{i=1}^{n} x^*(T_i x)\right)∥ \]

\[ \leq |α| ∥ f_1 ∥ \left|∑_{i=1}^{n} T_i\right| + |(1 - α)| ∥ f_2 ∥ ∥ \left∑_{i=1}^{n} T_i\right| \]

But ∥ f_1 ∥ = ∥ f_2 ∥ = 1, then

\[ \leq |α| \left|∑_{i=1}^{n} T_i\right| + |(1 - α)| \left|∑_{i=1}^{n} T_i\right| \]

\[ \leq |α| \left|∑_{i=1}^{n} T_i\right| + |(1 - α)| \left|∑_{i=1}^{n} T_i\right| - |α| \left|∑_{i=1}^{n} T_{i-1}\right| \]

\[ |φ(∑_{i=1}^{n} T_i)| \leq \{|∑_{i=1}^{n} T_i|\} \]

\[ |φ(∑_{i=1}^{n} T_i)| \leq ∥ φ ∥ ∥ |∑_{i=1}^{n} T_i| \leq |∑_{i=1}^{n} T_i| \]

\[ ∥ φ ∥ \left|∑_{i=1}^{n} T_{i-1}\right| \leq \left|∑_{i=1}^{n} T_i\right| \]

\[ \Rightarrow ∥ φ ∥ \leq 1 .................................................................(ii) \]

Therefore, (i) and (ii) imply that ∥ φ ∥ = 1

Since ∥ φ ∥ = 1, it follows that φ ∈ B(X^*). Hence φ(∑_{i=1}^{n} T_i) ∈ V(T_i)

⇒ V(T_i) is a convex set.

The Banach space numerical range for finite linear operators is associated with the numerical radius. Therefore, we give the definition of the numerical radius and present its elementary properties.

**Definition 2.1.5:**

Let X be a Banach space and T_i for i = 1,2,3,...,n, be linear operators. The Numerical radius for T_i is defined as the setNr(T_i) = Sup \{ |λ| : λ ∈ V(T_i) \}.

Equivalently, Nr(T_i) = Sup \{ |∑_{i=1}^{n} x^*(T_i x)| : x^* ∈ X^*, x ∈ X, x^*(x) = 1 \}.

**Proposition 2.1.6:**
Let $X$ be a Banach space and $T_i, S_i$ for $i = 1, 2, 3, \ldots, n$, be linear operators. Then the numerical radius for finite linear operators has the following elementary properties

i) $\text{Nr}(T_i + S_i) \leq \text{Nr}(T_i) + \text{Nr}(S_i)$

ii) $\text{Nr}(\lambda T_i) = |\lambda| \text{Nr}(T_i)$, where $\lambda$ is a scalar

iii) $\text{Nr}(U_i^{-1}T_iU_i) = \text{Nr}(T_i)$

iv) $\text{Nr}(T_i^*) = \text{Nr}(T_i)$

Proof

i) $\text{Nr}(T_i + S_i) = \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i + S_i)x| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i x + S_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i x) + \sum_{i=1}^{n} x^*(S_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$\leq \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\} + \text{Sup}\{|\sum_{i=1}^{n} x^*(S_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\}

\leq \text{Nr}(T_i) + \text{Nr}(S_i)$.

ii) $\text{Nr}(\lambda T_i) = \text{Sup}\{|\sum_{i=1}^{n} x^*(\lambda T_i)x| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\lambda||\sum_{i=1}^{n} x^*(T_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= |\lambda| \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i x)| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= |\lambda| \text{Nr}(T_i)$.

iii) $\text{Nr}(U_i^{-1}T_iU_i) = \text{Sup}\{|\sum_{i=1}^{n} x^*(U_i^{-1}T_iU_i)x| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i (x)U_i^{-1}U_i)| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(x)T_iU_i^{-1}U_i| : x^* \in X^*, x \in X, x^*(x) = 1\}$

Since $U_i$, is an isometry in the Banach space, then $U_i^{-1}U_i = I$, hence we obtain

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(x)T_i| : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i)x) : x^* \in X^*, x \in X, x^*(x) = 1\}$

Now, $x^*(x) = 1$, thus,

$= \text{Sup}\{|\sum_{i=1}^{n} x^*(T_i|x) : x^* \in X^*, x \in X, x^*(x) = 1\}$

$= \text{Nr}(T_i)$

iv) Let the Banach space $X$ be reflexive, $X^{**} = \{\hat{x} : x \in X\}$, where $\hat{x} = x^{**}$, then the numerical radius of operators $T_i^*$ is defined as the set

$\text{Nr}(T_i^*) = \text{Sup}\{|\sum_{i=1}^{n} \hat{x}T_i^*(x^*)| : x^* \in X^*, \hat{x} \in X^{**}, \hat{x}(x^*) = x^*(x) = 1\}$.

Now $X$ is a reflexive Banach space with $\hat{x}(x^*) = x^*(x)$, this shows that $T$ is a self-adjoint operator, that is, $T_i = T_i^*$. Hence, $\text{Nr}(T_i^*) = \text{Nr}(T_i)$.
Definition 2.1.7:

Let $X$ be a Banach space and $T_i$ for $i = 1, 2, 3, ..., n$, be linear operators. The spectrum for $T_i$ is the set $SP(T_i) = \{ \lambda \in \mathbb{C} : T_i - \lambda I \text{ is not invertible for } i = 1, 2, ..., n \}$

We strive to show that this spectrum is non-empty, closed and bounded. We also establish that the determined spectrum is contained in the closure of the numerical range for finite linear operators.

Definition 2.1.8:

The resolvent of $T_i$ is defined as $\mathbb{C} - SP(T_i)$

Proposition 2.1.9:

Let $X$ be a Banach space and $T_i$ for $i = 1, 2, 3, ..., n$, be linear operators and $SP(T_i)$ be the spectrum of $T_i$, then

i) $SP(T_i)$ is non-empty
ii) $SP(T_i)$ is bounded
iii) $SP(T_i)$ is closed
iv) $SP(T_i) \subseteq V(T_i)$

Proof

Proving i), ii) and iii), we will be equivalent to showing that $SP(T_i)$ is a non-empty compact set.

Let $I$ be an identity element in $X$ and for each $\lambda \in \mathbb{C}$ such that $|\lambda| > \|T_i\|$, then we have,

$$\frac{|\lambda|}{|\lambda|} > \frac{\|T_i\|}{|\lambda|},$$

i.e. $1 > \frac{\|T_i\|}{|\lambda|}$, thus $\|I - \frac{T_i}{|\lambda|}\| < 1$. This shows that $T_i - \lambda I$ is invertible and hence $\lambda \notin SP(T_i)$ with $\|\frac{T_i}{|\lambda|}\| < 1$. Furthermore, $SP(T_i)$ is bounded with $\|T_i\|$.

Remark 2.1.10:

Given $T_i, i = 1, 2, ..., n$ are linear operators, then $\|T_i\| = \left(\sum_{i=1}^{n}|T_i|^2\right)^{\frac{1}{2}}$.

Now, define a function $\phi_i : \mathbb{C} \to X$ by $\phi_i(\lambda) = T_i - \lambda I$ and denote the set of all invertible elements by $\mathcal{G}$. It is well known that the set of all invertible elements is open, and therefore $\phi_i^{-1}(\mathcal{G}) = \mathbb{C} - SP(T_i)$, that is, the resolvent set is open which implies that the spectrum $SP(T_i)$ is closed. Since $SP(T_i)$ is both bounded and closed, then it is a compact set.

Next, we show that $SP(T_i)$ is non-empty. We shall show this by contraction. Suppose that $SP(T_i)$ is empty, then $R(T_i) = -SP(T_i) = \mathbb{C}$. $R(T_i)$ is the entire analytic function on $\mathbb{C}$.

$$\|R(T_i)\lambda\| = \|(T_i - \lambda I)^{-1}\|$$
\[ \| \frac{1}{T_i - \lambda I} \| = \frac{1}{|\lambda|} \| \frac{1}{T_i \lambda^{-1} - I} \| \]

Taking limits as \( \lambda \to \infty \), we have
\[ \lim_{\lambda \to \infty} \| R(T_i) \lambda \| = \lim_{\lambda \to \infty} \left\{ \frac{1}{|\lambda|} \| \frac{1}{T_i \lambda^{-1} - I} \| \right\} \leq \frac{1}{|\lambda|} \lim_{\lambda \to \infty} \sup \left\{ \| \frac{1}{T_i \lambda^{-1} - I} \| \right\} \]
\[ \Rightarrow \lim_{\lambda \to \infty} \| R(T_i) \lambda \| = 0. \]

This shows that \( R(T_i) = 0 \) which is not true for \( 0 \) is not invertible. This contradicts our earlier supposition that \( SP(T_i) \) is empty and thus it is non-empty.

We now show that \( SP(T_i) \subseteq \overline{V(T_i)} \).

Let \( \lambda \in SP(T_i) \). Since \( X \) is complete, there exists a sequence \( \{x_n\} \) such that
\[ (T_i - \lambda)x_n \to 0 \text{ as } n \to \infty. \]
By the Hahn Banach theorem, there exists \( x_n^* \in X^* \) such that \( x_n^*(x_n) = 1 \forall n \in \mathbb{N} \).
Thus
\[ x_n^*(T_i x_n) \to \lambda \text{ as } n \to \infty \text{ and so } \lambda \in \overline{V(T_i)}. \]
Hence,
\[ SP(T_i) \subseteq \overline{V(T_i)}. \]

3. Conclusion

The numerical range has been studied extensively in the Hilbert spaces. This study focused on the numerical range for finite linear operators in the Banach spaces. In definition 2.3.1, we explicitly gave the definition of the numerical range for finite linear operators. We were able to give the properties of this numerical range and showed that they hold. In particular, we were able to prove that the Banach numerical range for finite linear operators is convex under the condition that the given Banach space is smooth in proposition 2.3.4. Associated with the numerical range is the numerical radius. The definition of the numerical radius associated with the Banach numerical range was given in definition 2.3.5 and its elementary properties were given in proposition 2.3.6. Finally, the spectrum for finite linear operators was established and its properties given in proposition 2.3.9. More importantly, it was established that the spectrum is contained in the closure of the numerical range.
References


