International Journal of Modern Mathematical Sciences, 2015, 13(2): 170-177

Modern Scientific Press | *International Journal of Modern Mathematical Sciences* | **ISSN:2166-286X**
Journal homepage:www.ModernScientificPress.com/Journals/ijmms.aspx | *Florida, USA*

Article

A Note on Bayesian Estimation of Inverse Weibull Distribution under LINEX and Quadratic Loss Functions

Sofi Mudasir, Afaq Ahmed and S.P Ahmad

Department of Statistics, University of Kashmir, Srinagar, India

* Author to whom correspondence should be addressed; E-Mail:baderaafaq@gmail.com

Article history: Received 28 January 2015, Received in revised form 25 March 2015, Accepted 20 April 2015, Published 6 May 2015.

Abstract: In this paper, the Bayes estimators of the scale parameter of inverse Weibull family of distributions have been derived by Quasi non-informative as well as conjugate priors under different scale-invariant loss functions, namely, LINEX loss function and Quadratic loss function. The risk functions of these estimators have been studied.

Keywords: Inverse Weibull distribution, Quasi prior and Gamma prior, LINEX loss function, Quadratic loss function.

Mathematics Subject Classification Code (2010): 68M15, 62F15

---

1. Introduction

The inverse Weibull distribution is another life time probability distribution which can be used in the reliability engineering discipline. The inverse Weibull distribution can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. It can also be used to determine the cost effectiveness, maintenance periods of reliability centered maintenance activities and applications in medicine, reliability and ecology. The inverse Weibull distribution provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of a constant tension, see Nelson (1982).The inverse Weibull distribution has initiated a large volume of research .For example, Calabria and Pulcini (1990) have discussed the maximum likelihood and least square estimations of its parameters, and Calabria and Pulcini (1994) have considered Bayes 2-sample prediction of the distribution. Keller (1985) obtained the inverse Weibull model by investigating failures of mechanical components subject to degradation. The two parameter exponentiated inverted Weibull distribution (EIWD) has been proposed by Flaih et al (2012). The two parameter inverse Weibull distribution has the following density function

$$F(x;\theta,\beta) = \exp\left(-\theta x^{-\beta}\right), x > 0 \qquad (1)$$

where θ>0 is the scale parameter and β>0 is the shape parameter. The density function corresponding to (1) is

$$f(x) = \theta\beta\ x^{-(\beta+1)}\left(e^{-\theta x^{-\beta}}\right)\ ;\ x > 0,\ \beta,\theta > 0 \qquad (2)$$

The aim of this paper is to propose the different methods of estimation of the parameters of the inverse Weibull distribution. In the next section, we obtain the MLE of the scale parameter θ in inverse Weibull when β is known. We also discuss the procedures to obtain the Bayes estimators for the unknown parameters using gamma prior, and Quasi prior under LINEX loss function and quadratic loss function.

2. Reliability Analysis

The reliability function $R(t)$, which is the probability of an item not failing prior to sometime t, is defined by $R(t) = 1 - F(t)$. The reliability function of IWD is given by

$$R(t,\theta,\beta) = 1-(e^{-\theta t^{-\beta}})$$

The other characteristic of interest of a random variable is the hazard rate function defined by

$$h(t) = \frac{f(t)}{1-F(t)}$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to time t. The hazard rate function for inverse random variable is given by

$$h(t,\theta,\beta) = \frac{\theta\beta t^{-(\beta+1)}(e^{-\theta t^{-\beta}})}{1-(e^{-\theta t^{-\beta}})}$$

3. Estimation of the Scale Parameter

3.1. Maximum Likelihood Estimator

If $x_1, x_2, ..., x_n$ is a random sample from exponentiated inverse Weibull distribution given by (2), then the likelihood function becomes:

$$L(\beta,\theta) = (\theta\beta)^n \prod_{i=1}^{n}\left(x_i\right)^{-(\beta+1)} \exp\left(-\theta\sum_{i=1}^{n} x_i^{-\beta}\right)$$

$$\log L(\beta,\theta) = n\log\theta + n\log\beta - (\beta+1)\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i^{-\beta}$$

As parameter β is known, the MLE of θ which maximize the log likelihood must satisfy the normal equation given by:

$$\frac{\partial \log L(\beta,\theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{-\beta} = 0$$

$$\Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i^{-\beta}} \qquad (3)$$

3.2. Bayes Estimator

We now derive the Bayes estimator of the parameter θ in IWD when the parameter β is known. We consider two different priors and two different loss functions.

(a)Quasi Prior: The first prior which we use is the Quasi prior. When there is no information about the distribution parameter, one may use Quasi prior as given by

$$\pi_1(\theta) = \frac{1}{\theta^d}; \theta > 0, d > 0$$

The quasi prior leads to diffuse prior when d=0 and to a non informative prior for a case when *d*=1.

(b) Gamma Prior: The second prior which we use is the Gamma prior. It is assumed that the scale parameter has a gamma prior distribution with shape and scale parameters c and d respectively, when it has the following pdf

$$\pi_2(\theta) = \frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-\theta d}; \theta > 0, c, d > 0$$

(i) LINEX loss function: The LINEX loss function is an asymmetric loss function which was introduced by Klebanov (1972) and used by Varian (1975) in the context of real estate assessment. Zellner (1986) and Varian (1975) have discussed its behavior and various applications.

The LINEX loss function is defined as

$$l(\theta,\hat{\theta}) = \exp\left\{b\left(\hat{\theta}-\theta\right)\right\} - b\left(\hat{\theta}-\theta\right) - 1$$

Where $b \neq 0$ determines the shape of the loss function.

(ii)Quadratic loss function: The use of quadratic loss function is common, for example when using least square techniques. It is often more mathematically tractable than other loss functions because of the properties of variances, as well as being symmetric: an error above the target causes the same loss as the same magnitude of error below the target.

If the target is t, then a quadratic loss function is

$$\lambda(x) = c(t - x)^2$$

For some constant c; the value of constant makes no difference to a decision and can be ignored by setting it equal to 1.

The Quadratic loss function is defined as

$$l(\theta, \hat{\theta}) = \left(\frac{\theta - \hat{\theta}}{\theta} \right)^2$$

where $\hat{\theta}$ is the estimate of θ.

3.3. Bayes Estimator under $\pi_1(\theta)$

Under $\pi_1(\theta)$, using (2), the posterior distribution is given by

$$P(\theta|x) = k\theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta} \right)$$

where k is independent of θ

and $k^{-1} = \int_0^\infty \theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta} \right) d\theta$

$$\Rightarrow k^{-1} = \frac{\Gamma(n-d+1)}{\left(\sum_{i=1}^{n} x_i^{-\beta} \right)^{n-d+1}}$$

$$\Rightarrow k = \frac{\left(\sum_{i=1}^{n} x_i^{-\beta} \right)^{n-d+1}}{\Gamma(n-d+1)}$$

Thus posterior distribution is given by

$$P(\theta|x) = \frac{\left(\sum_{i=1}^{n} x_i^{-\beta} \right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta} \right)$$

3.3.1. Estimator under LINEX loss function

By using LINEX loss function $l(\theta, \hat{\theta}) = \exp\{b(\hat{\theta} - \theta)\} - b(\hat{\theta} - \theta) - 1$, the risk function is given by

$$R(\hat{\theta}) = \int_0^\infty \left\{ \exp\left[b\left(\hat{\theta}-\theta\right)\right] - b\left(\hat{\theta}-\theta\right) - 1 \right\} \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta}\right) d\theta$$

$$R(\hat{\theta}) = \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \left\{ \exp\left[b\left(\hat{\theta}-\theta\right)\right] - b\left(\hat{\theta}-\theta\right) - 1 \right\} \theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta}\right) d\theta$$

$$R(\hat{\theta}) = \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}}{\left(b + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}} \exp(b\hat{\theta}) - b\hat{\theta} + \frac{b(n-d+1)}{\sum_{i=1}^{n} x_i^{-\beta}} - 1$$

Now solving $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, we obtain the Baye's estimator as

$$\hat{\theta}_l = \frac{1}{b} \log\left(\frac{\left(b + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}}{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}} \right) \tag{4}$$

3.3.2. Estimator under Quadratic loss function

By using Quadratic loss function $l(\theta, \hat{\theta}) = \left(\frac{\theta - \hat{\theta}}{\theta}\right)^2$, the risk function is given by

$$R(\hat{\theta}) = \int_0^\infty \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta}\right) d\theta$$

$$R(\hat{\theta}) = \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 \theta^{n-d} \exp\left(-\theta \sum_{i=1}^{n} x_i^{-\beta}\right) d\theta$$

$$R(\hat{\theta}) = 1 + \hat{\theta}^2 \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)^2}{(n-d)(n-d-1)} - 2\hat{\theta} \frac{\left(\sum_{i=1}^{n} x_i^{-\beta}\right)}{n-d}$$

Now solving $\frac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, we obtain the Baye's estimator as

$$\hat{\theta}_q = \frac{n-d-1}{\sum_{i=1}^{n} x_i^{-\beta}} \tag{5}$$

3.4. Bayes estimator under $\pi_2(\theta)$

Under $\pi_2(\theta)$, using (2), the posterior distribution is given by

$$P(\theta|x) = k\theta^{n+c-1}\exp\left[-\theta\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)\right]$$

Where k is independent of θ

and $k^{-1} = \int_0^\infty \theta^{n+c-1}\exp\left[-\theta\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)\right]d\theta$

$$\Rightarrow k^{-1} = \frac{\Gamma(n+c)}{\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}$$

$$\Rightarrow k = \frac{\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\Gamma(n+c)}$$

Thus posterior distribution is given by

$$P(\theta|x) = \frac{\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\Gamma(n+c)}\theta^{n+c-1}\exp\left[-\theta\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)\right]$$

3.4.1. Estimator under LINEX loss function

By using LINEX loss function $L(\theta,\hat{\theta}) = \exp\left[b\left(\hat{\theta}-\theta\right)\right]-b\left(\hat{\theta}-\theta\right)-1$, the risk function is given by

$$R(\hat{\theta}) = \int_0^\infty \left\{\exp\left[b\left(\hat{\theta}-\theta\right)\right]-b\left(\hat{\theta}-\theta\right)-1\right\}\frac{\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\Gamma(n+c)}\theta^{n+c-1}\exp\left[-\theta\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)\right]d\theta$$

$$R(\hat{\theta}) = \exp(b\hat{\theta})\frac{\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\left(b+d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}} - b\hat{\theta} + \frac{b(n+c)}{d+\sum_{i=1}^{n} x_i^{-\beta}} - 1$$

$$R(\hat{\theta}) = \frac{\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\Gamma(n+c)}\int_0^\infty \left\{\exp\left[b\left(\hat{\theta}-\theta\right)\right]-b\left(\hat{\theta}-\theta\right)-1\right\}\theta^{n+c-1}\exp\left[-\theta\left(d+\sum_{i=1}^{n} x_i^{-\beta}\right)\right]d\theta$$

Now solving $\dfrac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, we obtain the Baye's estimator as

$$\hat{\theta}_l = \frac{1}{b}\log\left(\frac{\left(b + d + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}\right) \tag{6}$$

3.4.2. Estimator under Quadratic loss function

By using quadratic loss function $l(\theta,\hat{\theta}) = \left(\dfrac{\theta - \hat{\theta}}{\theta}\right)^2$, the risk function is given by

$$R(\hat{\theta}) = \int_0^{\infty} \left(\frac{\theta - \hat{\theta}}{\theta}\right)^2 \frac{\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\Gamma(n+c)} \theta^{n+c-1} \exp\left[-\theta\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)\right] d\theta$$

$$R(\hat{\theta}) = \frac{\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)^{n+c}}{\Gamma(n+c)} \int_0^{\infty} \left(\frac{\theta - \hat{\theta}}{\theta}\right)^2 \theta^{n+c-1} \exp\left[-\theta\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)\right] d\theta$$

$$R(\hat{\theta}) = 1 + \hat{\theta}^2 \frac{\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)^2}{(n+c-1)(n+c-2)} - 2\hat{\theta}\frac{\left(d + \sum_{i=1}^{n} x_i^{-\beta}\right)}{(n+c-1)}$$

Now solving $\dfrac{\partial R(\hat{\theta})}{\partial \hat{\theta}} = 0$, we obtain the Baye's estimator as

$$\hat{\theta}_q = \frac{n+c-2}{d + \sum_{i=1}^{n} x_i^{-\beta}} \tag{7}$$

4. Conclusion

In this article, we have primarily studied the Bayes estimator of the scale parameter of the exponentiated Inverse Weibull distribution under Quasi and Gamma priors by assuming two different loss functions.

References

[1] Aljouharah, A., Estimating the parameters of an exponentiated inverted Weibull distribution under type-II censoring, *Applied Mathematical Sciences*, 7(2013): 1721-1736.

[2] Basu, A.P. and Ebrahimi, N., Bayesian approach to life testing and reliability estimation using asymmetric loss functions, *Journal of Statistical Planning and Inference*, 29(1992): 21-31.

[3] Calabria, R. and Pulcini, G., An engineering approach to Bayes estimation for the Weibull distribution, *Microelectronics Reliability*, 34(5)(1994): 789-802.

[4] Calabria, R. and Pulcini, G., An engineering approach to Bayes estimation for the Weibull distribution, *Microelectronics Reliability*, 34(1994): 789-802.

[5] Dey, D.K. and Liu, Pie-San L., On comparison of estimators in a generalized life model, *Micro-electron. Relia.*,45(1992): 207-221.

[6] Dey, D.K., Ghosh, M. and Srinivasan, C., Simultaneous estimation of parameters under entropy loss, *J. Statist.Plan.and Infer.*, (1987):347-363.

[7] El-Din, M.M., Riad, F.H. and El-Sayed, M.A., Statistical Inference and Prediction for the Inverse Weibull Distribution based on record data, *Journal of Statistical Applications and probability*, (2014):171-177.

[8] Flaih, A., Elsalloukh, H., Mendi, E. and Milanova, M., The Exponentiated Inverted Weibull Distribution, *Applied Mathematics & Information Sciences*, 6(2012): 167-171.

[9] Maswadah, M., Confidential confidence interval estimation for the inverse Weibull distribution based on censoring generalized order statistics, *journal of Statistical Computation and Simulation*, 73(2003): 887-898.

[10] Nelson, W (1982). Applied Life Data Analysis, J. Wiley, N. Y. USA.

[11] Soland, R.M., Bayesian analysis of the Weibull process with unknown scale parameter and its application to acceptance sampling, *IEEE Transactions on Reliability*, 17(1968): 84-90.

[12] Varian, H. R., *A Bayesian approach to real estate assessment*, Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage (eds: S.E. Fienberg and A.Zellner), North-Holland, Amsterdam,1975, pp.195-208.