New type of sequence space and matrix transformations

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Abstract: The main purpose of the present paper is to determine the necessary and sufficient conditions on a matrix sequence $\mathcal{A} = (A_v)$ in order that $\mathcal{A}$ belongs to the matrix class $(bv\,(u, p) : C)$ where $0 < p \leq \infty$.

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1. Preliminaries, Background and Notation

By ω, we denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. We write l_∞, c and c_0 for the spaces of all bounded, convergent and null sequences , respectively. Also by bs, cs, l_1 and l_p ,we denote the spaces of all bunded, convergent, absolutely and p-absolutely convergent series, respectively. We also denote by C and C_0 , the spaces of all convergent and null double sequences, respectively.

For the sequence spaces X and Y define the set $S(X : Y)$ by

$$S\,(X : Y) = \{\ z = (z_k) \in\ \omega\ : xz = (x_k z_k) \in Y\ \forall\ \ x \in\ \ X\ \}\,. \tag{1}$$

With the notation of (1) , $\alpha-$, $\beta-$ and $\gamma-$duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ , are defined by

$$X^\alpha = S(X : l_1),\ X^\beta = S(X : cs) \text{ and } X^\gamma = S(X : bs).$$

Let X and Y be two sequence spaces and let $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k \in \mathbb{N}$. Then, the matrix A defines the $A-$transformation from X *into* Y, if for every sequence $x=(x_k)\in X$ the sequence $Ax=((Ax)_n)$, the A-transform of x exists and is in Y; where $(Ax)_n=\sum_k a_{nk}x_k$. A sequence x is said to be A-summable to l if Ax converges to l if Ax-converges which is called as the A-limit of x. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A=\{\ x=(x_k)\in\omega\ :\ Ax\in X\ \}. \tag{2}$$

The approach of constructing a new sequence space by means of a particular method have been studied by several authors viz., (see, [1, 3-9]). The idea of $\mathcal{A}$-summability , was introduced by H.T. Bell in his doctoral work [2]. For $v=1,2,\ldots$, let $A_v=(a_{nk}(v))$ be an infinite matrix of real numbers and let $\mathcal{A}$ be a sequence of infinite matrices (A_v) and $X\subset\omega, Y\subset\omega$. Then, the matrix sequence $\mathcal{A}=(A_v)$ define the transformation from X into Y if every sequence $(x_k)\in X$ the double sequence $\mathcal{A}x=((\mathcal{A}x)_n^v)_{n,v=0}^{\infty}$, the $\mathcal{A}$-transform of x exists and is in Y ; where $(\mathcal{A}x)_n^v=\sum_k a_{nk}(v)x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X:Y)$, we denote the class of all such matrix sequences. A sequence x is said to be $\mathcal{A}$-summable to L if $(\mathcal{A}x)_n^v=L$ uniformly in v. We shall write throughout the paper for brevity that

$$\tilde{a}_{nk}(v)=\sum_{j=k}^{\infty}\frac{a_{nj}(v)}{u_k}$$

$$a(n,k,v)=\sum_{i=1}^{n}a_{ik}(v),$$

for all $n,k,v\in\mathbb{N}$. In [2], although no ordinary limitation method can correspond to almost convergence defined in [7], it is shown that this is possible using matrix sequences.

2. Main Results : The space $bv(u,p)$ of sequences of p-bounded variation was defined and studied by Basár, Altay and Mursaleen[1], where

$$bv\,(u,p) = \left\{ x = (x_k) \in \omega \ : \ \sum_k |u_k \triangle x_k|^p < \ \infty \right\} \ , \ (\ 0 < p \leq H < \infty).$$

It was proved that $bv(u,p)$ is a BK-space which is linearly isomorphic to the space $l(p)$ and the inclusion $bv\,(u,p) \supset l(p)$ strictly holds. The $\alpha-, \beta-$ *and* $\gamma-$duals of the space $bv(u,p)$ are determined. Define the sequence $y = (y_k)$, which will be frequently used, by the A^u-transform of a sequence $x = (x_k)\,,$ *i.e.,* $y_k = (u_k \triangle x_k)\ ,\quad k \in \mathbb{N}$.

We use the following Lemmas in proving the main results.

Lemma 2.1 [1]: The sequence space $bv(u,p)$ is linearly isomorphic to the space $l(p)$ *i, e.,* $bv(u,p) \cong l(p)$, where $0 < p_k \leq H < \infty$.

Lemma 2.2 [1]:.Define the sequence $b^{(k)}\,(u) = \left\{\ b_n^k(u)\right\}$ of the elements of the space $bv(u,p)$ for every fixed $k \in \mathbb{N}$ by

$$b_n^k\,(u) = \begin{cases} \frac{1}{u_k} \ , & n \geq k, \\ 0, & n < k. \end{cases} \tag{3}$$

Then the sequence $\left\{\ b_n^k(u)\right\}$ is a basis for the space $bv(u,p)$ and any $x \in bv(u,p)$has a unique representation of the form

$$x = \sum_k \lambda_k\,(u)\,b^k\,(u)\,, \tag{4}$$

where $\lambda_k\,(u) = (A^u x)_k$ for all $k \in \mathbb{N}$ and $0 < p \leq H < \infty$.

Theorem 2.3: Let $1 < p < \infty$. Then $\mathcal{A} \in (\ bv\,(u,p)\,, C\ )$ if and only if

$$\sup_{m,v} \sum_k \left| \sum_{j=k}^{m} a_{nj}(v) \right|^q < \infty\ (\ n \in \mathbb{N}\ ), \tag{5}$$

$$\sup_{n,v} \sum_k |\tilde{a}_{nk}(v)|^q < \infty, \tag{6}$$

$$\lim_n n\tilde{a}_{nk}\,(v) = \ \alpha_k\ (\text{ uniformly in } v\ )\ . \tag{7}$$

Proof : Let $\mathcal{A} \in (\ bv\,(u,p)\,,C \ )$ and $0 < p < \infty$. Then $\mathcal{A}x$ exists for every $x \in bv(u,p)$ and this implies that $\{a_{nk}(v)\} \in bv(u,p)^{\beta}$ for each $n, v \in \mathbb{N}$ which shows the necessity of (5).

Consider the following equation

$$\sum_k a_{nk}\,(v)\,x_k = \sum_k a_{nk}\,(v)\left(\sum_{j=0}^{k} \triangle x_j\right)$$
$$= \sum_j \sum_{k=j}^{\infty} a_{nk}(v) \frac{\triangle x_j}{u_j} u_j = \sum_j \widetilde{a}_{nk}(v) y_j.$$

That is , we have

$$\sum_k a_{nk}\,(v)\,x_k = \sum_j \tilde{a}_{nk}(v) y_j. \tag{8}$$

Taking supremum over $n, v$and applying Holder's inequality we obtain from (8) that

$$\sup_n \sum_k |a_{nk}\,(v)\,x_k| \leq \sup_n \left(\textstyle\sum_j |\tilde{a}_{nk}(v)|^q\right)^{\frac{1}{q}} \left(\textstyle\sum_k |y_k|^p\right)^{\frac{1}{p}} < \ \infty \ ,$$

there by proving the necessity of (6).

To prove the necessity of (7), consider , for every fixed $k \in \mathbb{N}$, the sequence of the elements of $bv(u,p)$ as

$$b_n^k(u) = \begin{cases} \frac{1}{u_k} & , \quad n \geq k, \\ 0 & , \quad n < k. \end{cases} \tag{9}$$

Since the $\mathcal{A}$-transform of $x \in bv(u,p)$ exists and lies in C by hypothesis, $\mathcal{A}b_n^{(k)} = \{\tilde{a}_{nk}(v)\}$ is also in C for every fixed $k \in \mathbb{N}$, which proves the necessity the (7).

Conversely, suppose that the conditions (5)-(7) holds and $x \in bv(u,p)$. Then $\mathcal{A}x$-exists. We observe for every $m, n \in \mathbb{N}$ that

$$\sum_{j=0}^{m}\left|\sum_{k=j}^{m}a_{nk}\left(v\right)x_{k}\right|\leq\max_{n,v}\sum_{j}\left|\tilde{a}_{nk}(v)y_{j}\right|$$

which leads us to the following fact, by letting $m, n \to \infty$ in (7) and using (5), we have

$$\sum_{j}\left|\sum_{k=j}^{\infty}\alpha_{j}\right|<\ \infty. \tag{10}$$

Hence, $(\alpha_k) \in bv(u,p)$ which implies that the series $\sum_k \alpha_k x_k$ is convergent for every $x \in bv\left(u,p\right)$.

Let us now consider the equality obtained from (8) with $a_{nk}\left(v\right)-\alpha_k$ instead of $a_{nk}\left(v\right)$, we see that

$$\sum_{k}\left[a_{nk}\left(v\right)-\alpha_{k}\right]x_{k}=\sum_{k}b_{nk}(v)y_{k} \tag{11}$$

where $\mathcal{B}=(b_{nk}(v))$ with $b_{nk}\left(v\right)=\sum_{j=k}^{\infty}a_{nk}\left(v\right)-\alpha_k$ for all $n,k,v\in\mathbb{N}$. Thus, we have at this stage from (9) with $\alpha_k=0$ for all $k\in\mathbb{N}$, that the matrix $\mathcal{B}$ belongs to the class $(l_p : c_0)$. Thus we see by (11) that

$$\lim_{n}\sum_{k}\left[a_{nk}\left(v\right)-\alpha_{k}\right]x_{k}=0\quad, \tag{12}$$

which means that $\mathcal{A}x\in C$ whenever $x\in bv(u,p)$ and this is what we wished to prove.

Note that for $p=\infty$ the condition for $\mathcal{A}\in\left(\ bv\left(u,p\right),C\ \right)$ are (6), (7) and

$$\sum_{k}\left|\sum_{j=k}^{m}a_{nj}(v)\ \right|<\infty.$$

The proof is similar to the above proof.

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