# New Exact Solutions of Some Nonlinear Partial Differential Equations via the Bernoulli Sub-ODE Method 

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#### Abstract

In this paper, we obtain new exact solutions of some nonlinear partial differential equations such as the ( $1+1$ )-dimensional travelling regularized long wave (TRLW) equation, the ( $2+1$ )-dimensional Calogero equation, the ( $3+1$ )-dimensional Jimbo-Miwa equation, and the variant shallow water wave equations via the Bernoulli subODE method.


Keywords: Nonlinear PDEs, Exact solutions, the Bernoulli Sub-ODE method.

## 1. Introduction

The nonlinear evolution equations (NLEEs) are widely used as models to describe complex physical phenomena in various field of science, particularly in fluid mechanics, solid state physics, plasma waves and chemical physics. Nonlinear equations covers also the following subjects: surface wave in compressible fluid, hydro magnetic waves in cold plasma, acoustic waves in harmonic crystal, ect.. The wide applicability of these equations is the main reason for they have attracted so much attention from mathematicians in the last decades. The investigation of the exact solutions of non linear partial differential equations (PDEs) plays an important role in the study of non-linear physical phenomena. When we want to understand the physical mechanism of phenomena in nature, described by non linear PDEs, exact solutions have to be explored. The study of nonlinear PDEs becomes one of
the most important topics in mathematical physics. Recently there are many new methods to obtain exact solutions of nonlinear PDEs such as, the tanh function method [5, 6, 28], the $\left({ }^{G} / G\right)$-expansion method [16, 17,19, 27, 29], the extended Jacobi elliptic function method [7, 9], the hyperbolic-sin function method [26], the Exp-function method [2, 3, 4, 11, 12, 13, 18, 30, 31], the improved Expfunction method [8, 21], and the generalized Bernoulli Sub-ODE method [10, 15, 25].

In this paper, we establish new exact solutions of some nonlinear partial differential equations (PDEs) of interest such as the ( $1+1$ )-dimensional travelling regularized long wave (TRLW) equation [20], the ( $2+1$ )-dimensional breaking soliton (Calogero) equation [1, 23, 24], the ( $3+1$ )-dimensional Jimbo-Miwa equation [22], and the variant shallow water wave equations [14] via the Bernoulli subODE method.

## 2. Description of the Bernoulli sub-ODE Method

Consider the following ordinary differential equation (ODE):
$G^{\prime}+\lambda G=\mu G^{2}$,
where $\lambda \neq 0, G=G(\xi)$.
When $\mu \neq 0$, Eq. (2.1) is the type of Bernoulli equation, and we can obtain the solution as
$G=\frac{1}{\frac{\mu}{\lambda}+d e^{\lambda \xi}}$,
where $d$ is an arbitrary constant.
When $\mu=0$, the solution of Eq. (2.1) is denoted by

$$
\begin{equation*}
G=d e^{-\lambda \xi} \tag{2.3}
\end{equation*}
$$

Suppose that a nonlinear equation, say in two or three independent variables $x, y$, and $t$ is given by

$$
\begin{equation*}
P\left(u, u_{t}, u_{x}, u_{y}, u_{t t}, u_{x t}, u_{y t}, u_{x x}, u_{y y}, \ldots\right)=0 \tag{2.4}
\end{equation*}
$$

where $u=u(x, y, t)$ is an unknown function, $P$ is a polynomial in $u=u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 1. We suppose that

$$
\begin{equation*}
u(x, y, t)=u(\xi), \quad \xi=\xi(x, y, t) \tag{2.5}
\end{equation*}
$$

The travelling wave variable (2.5) permits us reducing Eq. (2.4) to an ODE for $u=u(\xi)$

$$
\begin{equation*}
P\left(u, u^{\prime}, u^{\prime \prime}, \ldots\right)=0 \tag{2.6}
\end{equation*}
$$

Step 2. Suppose that the solution of (2.6) can be expressed by a polynomial in $G$ as follows:

$$
\begin{equation*}
u(\xi)=\alpha_{m} G^{m}+\alpha_{m-1} G^{m-1}+\ldots \tag{2.7}
\end{equation*}
$$

where $G=G(\xi)$ satisfies Eq. (2.1), and $\alpha_{m}, \alpha_{m-1}, \ldots$ are constants to be determined later, $\alpha_{m} \neq 0$. The positive integer $m$ can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.6).

Step 3. Substituting (2.7) into (2.6) and using (2.1), collecting all terms with the same order of $G$ together, the left-hand side of Eq. (2.6) is converted into another polynomial in $G$.
Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_{m}, \alpha_{m-1}, \ldots, \lambda, \mu$.

Step 4. Solving the algebraic equations system in step 3, and by using the solutions of Eq. (2.1), we can construct the travelling wave solutions of the nonlinear equation (2.6).

## 3. Applications

In order to illustrate the effectiveness of the Bernoulli sub-ODE method, examples of mathematical interests are chosen as follows:

### 3.1. The (1+1)-dimensional Travelling Regularized long (TRLW) Wave Equation

In this section, we consider the TRLW equation
$u_{t}+u_{x}+\beta u u_{x}+u_{x t t}=0$

Suppose that
$u(x, t)=u(\xi)=k(x-c t)$,
where $k, c$ are constants that to be determined later.
By Eq. (3.1.2) and Eq. (3.1.1) converted into ODE
$k(1-c) u^{\prime}+\beta k u u^{\prime}+k^{3} c^{2} u^{\prime \prime \prime}=0$
Suppose that the solution of (3.1.3) can be expressed by a polynomial in $G$ as follows:
$u(\xi)=\sum_{i=0}^{m} a_{i} G^{i}$,
where $a_{i}$ are constants, and $\boldsymbol{G}=\boldsymbol{G}(\xi)$ satisfies Eq. (2.1). Balancing the order of $\boldsymbol{u} \boldsymbol{u}^{\prime}$ and $u^{\prime \prime \prime}$ in Eq. (3.1.3), we obtain $m=2$. So Eq. (3.1.4) can be rewritten as
$u(\xi)=a_{2} G^{2}+a_{1} G+a_{0}, a_{2} \neq 0$
where $a_{0}, a_{1}$ are constants to be determined later.

Substituting Eq. (3.1.5) into Eq. (3.1.3) and collecting all the terms with the same power of $G$ together, the left-hand side of Eq. (3.1.3) is converted into another polynomial in $G$. Equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$
\begin{aligned}
G^{1}: & -k \lambda\left(\beta a_{1} a_{0}+a_{1}+c^{2} k^{2} a_{1} \lambda^{2}-c a_{1}\right)=0 \\
G^{2}: & -k \lambda\left(2 a_{2}-6 c^{2} k^{2} a_{1} \mu \lambda-2 c a_{2}+\beta a_{1}^{2}+2 \beta a_{2} a_{0}+8 c^{2} k^{2} a_{2} \lambda^{2}\right) \\
\quad & +k \mu\left(\beta a_{1} a_{0}+a_{1}+c^{2} k^{2} a_{1} \lambda^{2}-c a_{1}\right)=0 \\
G^{3}: & -\lambda k\left(6 c^{2} k^{2} a_{1} \mu^{2}+3 \beta a_{2} a_{1}-30 c^{2} k^{2} a_{2} \mu \lambda\right)+ \\
& k \mu\left(2 a_{2}-6 c^{2} k^{2} a_{1} \mu \lambda-2 c a_{2}+\beta a_{1}^{2}+2 \beta a_{2} a_{0}+8 c^{2} k^{2} a_{2} \lambda^{2}\right)=0 \\
G^{4}: & -k \lambda\left(2 \beta a_{2}^{2}+24 c^{2} k^{2} a_{2} \mu^{2}\right)+ \\
& k \mu\left(6 c^{2} k^{2} a_{1} \mu^{2}+3 \beta a_{2} a_{1}-30 c^{2} k^{2} a_{2} \mu \lambda\right)=0 \\
G^{5}: & k \mu\left(2 \beta a_{2}^{2}+24 c^{2} k^{2} a_{2} \mu^{2}\right)=0
\end{aligned}
$$

Solving the algebraic equations above, yields:
$k=k, \quad \lambda=\lambda, \quad a_{0}=-\frac{1+c^{2} k^{2} \lambda^{2}-c}{\beta}, \quad a_{1}=\frac{12 \lambda c^{2} k^{2} \mu}{\beta}$,
$\mu=\mu, \quad a_{2}=\frac{-12 \mu^{2} c^{2} k^{2}}{\beta}$

Substituting (3.1.6) into (3.1.5), we have
$u(\xi)=-\frac{1+c^{2} k^{2} \lambda^{2}-c}{\beta}+\frac{12 \lambda c^{2} k^{2} \mu}{\beta} G-\frac{12 \mu^{2} c^{2} k^{2}}{\beta} G^{2}$

Combining with Eq. (2.2), we can obtain the traveling wave solutions of Eq. (3.1.1) as follows:

$$
\begin{equation*}
u(x, t)=-\frac{1+c^{2} k^{2} \lambda^{2}-c}{\beta}+\frac{12 \lambda c^{2} k^{2} \mu}{\beta\left(\frac{\mu}{\lambda}+d e^{\lambda k(x-c t)}\right)}-\frac{12 \mu^{2} c^{2} k^{2}}{\beta\left(\frac{\mu}{\lambda}+d e^{\lambda k(x-c t)}\right)^{2}} \tag{3.1.8}
\end{equation*}
$$



Figure 1: Traveling wave solution of Eq. (3.1.1) for solution 3.1.8 when $k=c=\mu=\lambda=1, \quad \beta=2$.

### 3.2. The (2+1)-dimensional Breaking Soliton (Calogero) Equation

In this section, we consider the (2+1)-dimensional breaking soliton equation:
$u_{x t}-4 u_{x} u_{x y}-2 u_{y} u_{x x}+u_{x x y y}=0$
Suppose that
$u(x, y, t)=u(\xi)=k x+L y+\omega t$,
where $k, L, \omega$ are constants that to be determined later.
Substitute Eq. (3.2.2) into Eq. (3.2.1) and integrate the result with respect to $\xi$, Eq. (3.2.1) converted into ODE
$k \omega u^{\prime}-3 k^{2} L\left(u^{\prime}\right)^{2}+k^{3} L u^{\prime \prime \prime}=g$,
where $g$ is the integration constant.
Suppose that the solution of (3.2.3) can be expressed by a polynomial in $G$ as follows:
$u(\xi)=\sum_{i=0}^{m} a_{i} G^{i}$,
where $a_{i}$ are constants, and $\boldsymbol{G}=\boldsymbol{G}(\xi)$ satisfies Eq. (2.1). Balancing the order of $\left(u^{\prime}\right)^{2}$ and $u^{\prime \prime \prime}$ in Eq. (3.2.3), we obtain $m=1$. S0 Eq. (3.2.4) can be rewritten as

$$
\begin{equation*}
u(\xi)=a_{1} G+a_{0}, \quad a_{1} \neq 0 \tag{3.2.5}
\end{equation*}
$$

where $a_{0}$ is a constant to be determined later.
Substituting (3.2.5) into (3.2.3) and collecting all the terms with the same power of $G$ together, the left-hand side of Eq. (3.2.3) is converted into another polynomial in $G$. Equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:
$G^{0}:-g=0$
$G^{1}:-k^{3} L a_{1} \lambda^{3}-k \omega a_{1} \lambda=0$
$G^{2}: 7 k^{3} L a_{1} \mu \lambda^{2}+k \omega a_{1} \mu-3 k^{2} L a_{1}^{2} \lambda^{2}=0$
$G^{3}=-12 k^{3} L a_{1} \mu^{2} \lambda+6 k^{2} L a_{1}^{2} \mu \lambda=0$
$G^{4}:-3 k^{2} L a_{1}^{2} \mu^{2}+6 k^{3} L a_{1} \mu^{3}=0$
Solving the algebraic equations above, yields:

$$
\begin{align*}
& g=0, \quad a_{1}=2 k \mu, \quad k=k, \quad \mu=\mu, \quad \lambda=\lambda \\
& L=L, \quad \omega=-k^{2} \lambda^{2} L, \quad a_{0}=a_{0} \tag{3.2.6}
\end{align*}
$$

Substituting Eq. (3.2.6) into Eq. (3.2.5), we have

$$
\begin{equation*}
u(\xi)=a_{0}+2 k \mu G \tag{3.2.7}
\end{equation*}
$$

Combining with Eq. (2.2), we can obtain the traveling wave solutions of Eq. (3.2.1) as follows:

$$
\begin{equation*}
u(x, y, t)=a_{0}+\frac{2 k \mu}{\frac{\mu}{\lambda}+d e^{\lambda\left(k x+L y-k^{2} \lambda^{2} L t\right)}} \tag{3.2.8}
\end{equation*}
$$



Figure 2: Traveling wave solution of Eq. 3.2.1 for solution 3.2.8 when $a_{n}=3, \mu=d=2, \lambda=L=k=1, y=-1$

### 3.3. The (3+1)-dimensional Jimbo-Miwa Equation

In this section, we consider the (3+1)-dimensional Jimbo-Miwa equation:

$$
\begin{equation*}
u_{x x x y}+3 u_{y} u_{x x}+3 u_{x} u_{x y}+2 u_{y t}-3 u_{x z}=0 \tag{3.3.1}
\end{equation*}
$$

Suppose that
$u(x, y, z, t)=u(\xi)=k x+L y+m z+\omega t$,
where $k, L, m, \omega$ are constants that to be determined later.
Substitute Eq. (3.3.2) into Eq. (3.3.1) and integrate the result with respect to $\xi$, Eq. (3.3.1) converted into ODE
$k^{3} L u^{\prime \prime \prime}+3 k^{2} L\left(u^{\prime}\right)^{2}+(2 L \omega-3 k m) u^{\prime}=g$,
where $g$ is the integration constant.
Suppose that the solution of (3.3.3) can be expressed by a polynomial in $G$ as follows:
$u(\xi)=\sum_{i=0}^{m} a_{i} G^{i}$,
where $a_{i}$ are constants, and $G=\boldsymbol{G}(\xi)$ satisfies Eq. (2.1) Balancing the order of $\left(u^{\prime}\right)^{2}$ and $u^{\prime \prime \prime}$ in Eq. (3.33), we obtain $m=1$. So Eq. (3.3.4) can be rewritten as
$u(\xi)=a_{1} G+a_{0}, a_{1} \neq 0$
where $a_{0}$ is a constant to be determined later.
Substituting Eq. (3.3.5) into Eq. (3.3.3) and collecting all the terms with the same power of $G$ together, the left-hand side of Eq. (3.3.3) is converted into another polynomial in G. Equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:
$G^{0}:-g=0$
$G^{1}:-k^{3} L a_{1} \lambda^{3}+3 a_{1} k m \lambda-2 a_{1} L \omega \lambda=0$
$G^{2}:-3 a_{1} k m \mu+3 k^{2} L a_{1}^{2} \lambda^{2}+7 k^{3} L a_{1} \lambda^{2} \mu+2 a_{1} L \omega \mu=0$
$G^{3}=-12 k^{3} L a_{1} \lambda \mu^{2}-6 k^{2} L a_{1}^{2} \lambda \mu=0$
$G^{4}: 3 k^{2} L a_{1}^{2} \mu^{2}+6 k^{3} L a_{1} \mu^{3}=0$
Solving the algebraic equations above, yields:
$g=0, \quad a_{1}=-2 k \mu, \quad k=k, \quad \mu=\mu, \quad \lambda=\lambda$,
$L=L, \quad \omega=\omega, \quad m=\frac{L\left(k^{3} \lambda^{2}+2 \omega\right)}{3 k}, \quad a_{0}=a_{0}$
Substituting Eq. (3.3.6) into Eq. (3.3.5), we have

$$
\begin{equation*}
u(\xi)=a_{0}-2 k \mu G \tag{3.3.7}
\end{equation*}
$$

Combining with Eq. (2.2), we can obtain the traveling wave solutions of Eq. (3.3.1) as follows:
$u(x, y, z, t)=a_{0}-\frac{2 k \mu}{\frac{\mu}{\lambda}+d e^{\lambda\left(k x+L y+\frac{L\left(k^{3} \lambda^{2}+2 \omega\right) z}{3 k}+\omega t\right)}}$


Figure 3: Traveling wave solution of Eq. (3.3.1) for solution 3.3.8 when $a_{n}=10, \mu=\lambda=k=L=1, d=5$, $\omega=2, y=z=-1$

### 3.4. The Variant Shallow Water Wave Equations

In this section, we consider the variant shallow water wave equations
$u_{t}+v_{x}+u u_{x}-\varepsilon^{2} u_{x x t}=0$,
$v_{t}+u_{x}+(u v)_{x}+\delta^{2} u_{x x x}=0$
Suppose that
$u(x, t)=u(\xi), \quad v(x, t)=v(\xi), \quad \xi=k(x-c t)$,
where $k, c$ are constants that to be determined later.
By Eq. (3.4.3), Eq. (3.4.1) and Eq. (3.4.2) converted into ODE
$-k c u^{\prime}+k v^{\prime}+k u u^{\prime}+\varepsilon^{2} k^{3} c u^{\prime \prime \prime}=0$,
$-k c v^{\prime}+k u^{\prime}+k u^{\prime} v+k v^{\prime} u+\delta^{2} k^{3} u^{\prime \prime \prime}=0$
Suppose that the solution of Eq. (3.4.4) and Eq. (3.4.5) can be expressed by a polynomial in $G$ as follows:
$u(\xi)=\sum_{i=0}^{m} a_{i} G^{i}$,
$\nu(\xi)=\sum_{i=0}^{m} b_{i} G^{i}$
where $a_{i}$ and $b_{i}$ are constants, and $\boldsymbol{G}=\boldsymbol{G}(\xi)$ satisfies Eq. (2.1). Balancing the order of $\boldsymbol{u} \boldsymbol{u}^{\prime}$ and $u^{\prime \prime \prime}$ in Eq. (3.4.4) and the order of $v u^{\prime}$ and $u^{\prime \prime \prime}$ in Eq. (3.4.5), we obtain $m=n=2$. So Eq. (3.4.6) and Eq. (3.4.7) can be rewritten respectively as
$u(\xi)=a_{2} G^{2}+a_{1} G+a_{0}, a_{2} \neq 0$
$v(\xi)=b_{2} G^{2}+b_{1} G+b_{0}, b_{2} \neq 0$
where $a_{i}, b_{i}$ are constants to be determined later.
Substituting Eq. (3.4.6) and Eq. (3.4.7) into Eq. (3.4.4) and Eq. (3.4.5), collecting all the terms with the same power of $G$ together, the left-hand side of Eq. (3.4.4) and Eq. (3.4.5) are converted into other polynomials in $G$. Equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq. (3.4.4):
$G^{1}:-k \lambda\left(-c a_{1}+b_{1}+\varepsilon^{2} k^{2} c a_{1} \lambda^{2}+a_{0} a_{1}\right)=0$
$G^{2}:-k \lambda\left(-2 c a_{2}+2 a_{0} a_{2}+a_{1}^{2}+2 b_{2}-6 \varepsilon^{2} k^{2} c a_{1} \mu \lambda+8 \varepsilon^{2} k^{2} c a_{2} \lambda^{2}\right)$
$+k \mu\left(-c a_{1}+b_{1}+\varepsilon^{2} k^{2} c a_{1} \lambda^{2}+a_{0} a_{1}\right)=0$
$G^{3}:-k \lambda\left(6 \varepsilon^{2} k^{2} c a_{1} \mu^{2}+3 a_{1} a_{2}-30 \varepsilon^{2} k^{2} c a_{2} \mu \lambda\right)$
$+k \mu\left(-2 c a_{2}+2 a_{0} a_{2}+a_{1}^{2}+2 b_{2}-6 \varepsilon^{2} k^{2} c a_{1} \mu \lambda+8 \varepsilon^{2} k^{2} c a_{2} \lambda^{2}\right)=0$
$G^{4}:-k \lambda\left(2 a_{2}^{2}+24 \varepsilon^{2} k^{2} c a_{2} \mu^{2}\right)+$
$k \mu\left(6 \varepsilon^{2} k^{2} c a_{1} \mu^{2}+3 a_{1} a_{2}-30 \varepsilon^{2} k^{2} c a_{2} \mu \lambda\right)=0$
$G^{5}: k \mu\left(2 a_{2}{ }_{2}+24 \varepsilon^{2} k^{2} c a_{2} \mu^{2}\right)=0$
For Eq. (3.4.5):

$$
\begin{aligned}
G^{1}: & -k \lambda\left(-c b_{1}+\delta^{2} k^{2} a_{1} \lambda^{2}+a_{1}+a_{1} b_{0}+b_{1} a_{0}\right)=0 \\
G^{2}: & -k \lambda\left(2 a_{2}+2 a_{1} b_{1}+2 a_{2} b_{0}-6 \delta^{2} k^{2} a_{1} \mu \lambda-2 c b_{2}+2 b_{2} a_{0}+8 \delta^{2} k^{2} a_{2} \lambda^{2}\right) \\
& +k \mu\left(-c b_{1}+\delta^{2} k^{2} a_{1} \lambda^{2}+a_{1}+a_{1} b_{0}+b_{1} a_{0}\right)=0 \\
G^{3}: & -k \lambda\left(3 a_{2} b_{1}+6 \delta^{2} k^{2} a_{1} \mu \lambda+3 a_{1} b_{2}-30 \delta^{2} k^{2} a_{2} \mu \lambda\right) \\
& +k \mu\left(2 a_{2}+2 a_{1} b_{1}+2 a_{2} b_{0}-6 \delta^{2} k^{2} a_{1} \mu \lambda-2 c b_{2}+2 b_{2} a_{0}+8 \delta^{2} k^{2} a_{2} \lambda^{2}\right)=0 \\
G^{4}: & -k \lambda\left(24 \delta^{2} k^{2} a_{2} \mu^{2}+4 a_{2} b_{2}\right)+ \\
& k \mu\left(3 a_{2} b_{1}+6 \delta^{2} k^{2} a_{1} \mu \lambda+3 a_{1} b_{2}-30 \delta^{2} k^{2} a_{2} \mu \lambda\right)=0 \\
G^{5}: & k \mu\left(24 \delta^{2} k^{2} a_{2} \mu^{2}+4 a_{2} b_{2}\right)=0
\end{aligned}
$$

Solving the algebraic equations above, yields:
$a_{0}=-\frac{\delta^{2}-2 c^{2} \varepsilon^{2}+2 \varepsilon^{4} k^{2} c^{2} \lambda^{2}}{2 \varepsilon^{2} c}, \quad b_{0}=-\frac{2 \delta^{2} k^{2} \lambda^{2} \varepsilon^{4} c^{2}+4 \varepsilon^{4} c^{2}-\delta^{4}}{4 \varepsilon^{4} c^{2}}$,
$a_{1}=12 \lambda \varepsilon^{2} k^{2} c \mu, \quad, b_{1}=6 k^{2} \lambda \delta^{2} \mu, \quad a_{2}=-12 \varepsilon^{2} k^{2} c \mu^{2}, \quad, b_{2}=-6 \delta^{2} k^{2} \mu^{2}$,
$k=k, \quad \lambda=\lambda, \quad \mu=\mu, \quad c=c$

Substituting Eq. (3.4.8) into Eq. (3.4.6) and Eq. (3.4.7), we have

$$
\begin{equation*}
u(\xi)=-\frac{\delta^{2}-2 c^{2} \varepsilon^{2}+2 \varepsilon^{4} k^{2} c^{2} \lambda^{2}}{2 \varepsilon^{2} c}+12 \lambda \varepsilon^{2} k^{2} c \mu G-12 \varepsilon^{2} k^{2} c \mu^{2} G^{2} \tag{3.4.9}
\end{equation*}
$$

$v(\xi)=-\frac{2 \delta^{2} k^{2} \lambda^{2} \varepsilon^{4} c^{2}+4 \varepsilon^{4} c^{2}-\delta^{4}}{4 \varepsilon^{4} c^{2}}+6 k^{2} \lambda \delta^{2} \mu G-6 \delta^{2} k^{2} \mu^{2} G^{2}$

Combining with Eq. (2.2), we have:

$$
\begin{equation*}
u(x, t)=-\frac{\delta^{2}-2 c^{2} \varepsilon^{2}+2 \varepsilon^{4} k^{2} c^{2} \lambda^{2}}{2 \varepsilon^{2} c}+\frac{12 \lambda \varepsilon^{2} k^{2} c \mu}{\frac{\mu}{\lambda}+d e^{\lambda k(x-c t)}}-\frac{12 \varepsilon^{2} k^{2} c \mu^{2}}{\left(\frac{\mu}{\lambda}+d e^{\lambda k(x-c t)}\right)^{2}} \tag{3.4.11}
\end{equation*}
$$

$v(x, t)=-\frac{2 \delta^{2} k^{2} \lambda^{2} \varepsilon^{4} c^{2}+4 \varepsilon^{4} c^{2}-\delta^{4}}{4 \varepsilon^{4} c^{2}}+\frac{6 k^{2} \lambda \delta^{2} \mu}{\frac{\mu}{\lambda}+d e^{\lambda k(x-c t)}}-\frac{6 \delta^{2} k^{2} \mu^{2}}{\left(\frac{\mu}{\lambda}+d e^{\lambda k(x-c t)}\right)^{2}}$


Figure 4: $\mu(x, t)$, when $\delta=\varepsilon=c=\mu=1, \lambda=k=d=2$


Figure 5: $v(x, t)$, when $\delta=\varepsilon=c=\mu=1, \lambda=k=d=2$

## 4. Conclusion

In this paper, the Bernoulli sub-ODE method has been successfully applied to obtain new solutions of some nonlinear partial differential equations. Thus, the Bernoulli sub-ODE method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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