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Article

Generalized h-Randers Change of Finsler Metric

H.S. Shukla, O.P. Pandey *and Neelam Mishra

Department of Mathematics & Statistics, D.D.U Gorakhpur University, Gorakhpur, India

* Author to whom correspondence should be addressed; Email: profhsshuklagkp@rediffmail.com, oppandey1988@gmail.com, pneelammishra@gmail.com

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a generalized h-Randers change of Finsler metric becomes a projective change .We have also found a condition under which a generalized h-Randers change of Douglas space becomes a Douglas space.

Keyword: Randers change, generalized Randers change, h-vector, Finsler space, projective change.

Mathematics Subject Classification (2010): 53B40, 53C60

1. Introduction

Let $F^n = (M^n,L)$ be a n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function L(x,y).Various changes of Finsler metric have been studied recently papers [2],[3],[4],[5],[6] and [7].

The necessary and sufficient conditions for these changes to be projective have been obtained. The conditions for Douglas spaces with the changed metric to remain Douglas spaces have been found out.

The generalized h-Randers change of Finsler metric is given by

(1.1) $$\overline{L}(x,y) = (L^m + \beta^m)^{1/m} \quad \text{where } \beta(x,y) = b_i(x,y)\, y^i$$

and $b_i(x,y)$ in the transformation (1.1) is an h-vector, so that $\frac{\partial b_i}{\partial y^j}$ is proportional to the angular metric tensor h_{ij}.

Let

(1.2) $$\frac{\partial b_i}{\partial y^j} = \rho h_{ij},$$

where ρ is any scalar function of x ,y and $h_{ij} = g_{ij} - l_i l_j$.

It has been show by Shukla, Pandey and Joshi in [8] that

(1.3) $$\dot{\partial}_k \rho = -\frac{\rho}{L} l_k \quad , \text{for } n > 2 \ , \qquad \text{where} \quad \dot{\partial}_k \equiv \frac{\partial}{\partial y^k}.$$

We shall use the equation (1.3) without quoting it in the present paper.

Let $\beta = b_i(x,y)y^i$ be defined throughout the manifold M^n. Then $L \rightarrow (L^m + \beta^m)^{1/m}$ is called generalized h-Randers change of Finsler metric. If we write $\overline{L} = (L^m + \beta^m)^{1/m}$ and $\overline{F}^n = (M^n, \overline{L})$ then the Finsler space $\overline{F}^n$ is said to beobtained from F^n by a generalized h- Randers change of Finsler metric .The quantities corresponding to $\overline{F}^n$ will be denoted by putting bar over those quantities .

The fundamental quantities of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial yi\, \partial y^j}\ , \quad l_i = \frac{\partial L}{\partial y^i} \text{and} \quad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j$$

We shall denote the partial derivatives with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L\ , L_{ij} = \dot{\partial}_i \dot{\partial}_j L\ , L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L\ .$$

Then $L_i = l_i \ , \quad L^{-1} h_{ij} = L_{ij}$

The geodesics of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2\, G^i(x, \frac{dx}{ds}) = 0,$$

where $G^i(x,y)$ are positively homogeneous of degree two and are given by

$$2G^i = g^{ij}\left(y^r \dot{\partial}_j \partial_r F - \partial_j F\right), \quad F = \frac{L^2}{2}$$

where g^{ij} are the inverse of g_{ij} .

Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of Finsler space is given by [10]

$$G_j^i = \frac{\partial G^i}{\partial y^j}, \quad G_{jk}^i = \frac{\partial G_j^i}{\partial y^k}$$

The Cartan's connection $C\Gamma = (F_{jk}^i, G_j^i, G_{jk}^i)$ is constructed from L with the help of following axioms [10]:

(1) Cartan's connection CΓ is v-metrical.

(2) Cartan's connection CΓ is h-metrical.

(3) The h(v) -torsion tensor field S of Cartan's connection vanishes.

(4) The h(h)-torsion tensor field T of Cartan's connection vanishes.

(5) The deflection tensor field D of Cartan's connection vanishes.

The h- and v - covariant derivatives with respect to Cartan's connection are denoted by $|_k$ and $|_k$ respectively. It is clear that the h-covariant derivative of L with respect to BΓ and CΓ is the same and vanishes identically .Furthermore, the h-covariant derivatives of L_i, L_{ij} with respect to CΓ are also zero .We shall write

$$2r_{ij} = b_{i|j} + b_{j|i} \quad , \quad 2s_{ij} = b_{i|j} - b_{j|i}$$

2. Difference Tensor of Generalized h-Randers Change

The generalized h-Randers change of Finsler metric L is given by (1.1)

(2.1) $\overline{G}^i = G^i + D^i$.

Then $\overline{G}_j^i = G_j^i + D_j^i$ and $\overline{G}_{jk}^i = G_{jk}^i + D_{jk}^i$, were $D_j^i = \dot{\partial}_j D^i$ and $D_{jk}^i = \dot{\partial}_k D_j^i$.

The tensors D^i, D_j^i *and* D_{jk}^i are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i , we deal with equation $L_{ij|k} = 0$[9], i.e.,

(2.2) $\partial_k L_{ij} - L_{ijk} G_k^r - L_{rj} F_{ik}^r - L_{ir} F_{jk}^r = 0$.

Since $\dot{\partial}_i \beta = b_i$, from (1.1), we have

(2.3)

(a) $\overline{L_i} = p\, L_i + q\, b_i$,

where $p = \frac{L^{m-1}}{(L^m+\beta^m)^{(m-1)/m}}$ and $q = \frac{\beta^{m-1}}{(L^m+\beta^m)^{(m-1)/m}}$

(b) $\overline{L}_{ij} = \mu\, L_{ij} + \gamma\{\beta^2 L_i L_j - L\beta(L_i b_j + L_j b_i) + L^2 b_i b_j\}$, where $\mu = \frac{L^{m-1}+\rho L\beta^{m-1}}{(L^m+\beta^m)^{(m-1)/m}}$ and $\gamma = \frac{(m-1)(L\beta)^{m-2}}{(L^m+\beta^m)^{(2m-1)/m}}$,

(c) $\partial_j\overline{L}_i = \beta\gamma(\beta L_i - Lb_i)\partial_j L + L\gamma(Lb_i - \beta L_i)\partial_j\beta + p\partial_j L_i + q\partial_j b_i$,

(d) $\partial_k\overline{L}_{ij} = \mu\partial_k L_{ij} + \{[\rho q + \beta\gamma(\beta - \rho L^2)]L_{ij} + \eta L_i L_j - \xi(L_i b_j + L_j b_i) + \varphi b_i b_j\}\partial_k L + \{L\gamma(\rho L^2 - \beta)L_{ij} - \xi L_i L_j + \varphi(L_i b_j + L_j b_i) + \delta b_i b_j\}\partial_k\beta + \beta\gamma(\beta L_j - Lb_j)\partial_k L_i + \beta\gamma(\beta L_i - Lb_i)\partial_k L_j + L\gamma(Lb_j - \beta L_j)\partial_k b_i + L\gamma(Lb_i - \beta L_i)\partial_k b_j + LqL_{ij}\partial_k\rho$,

where

$$\eta = (m-1)\beta^m L^{m-3}[(m-2)\beta^m - (m+1)L^m](L^m + \beta^m)^{(1-3m)/m},$$

$$\xi = \beta(m-1)(L\beta)^{m-2}[m(\beta^m - L^m) - \beta^m](L^m + \beta^m)^{(1-3m)/m},$$

$$\varphi = L(m-1)(L\beta)^{m-2}[m(\beta^m - L^m) + L^m](L^m+\beta^m)^{(1-3m)/m},$$

$$\delta = (m-1)L^m\beta^{m-3}[(m-2)L^m - (m+1)\beta^m](L^m + \beta^m)^{(1-3m)/m},$$

(e) $\overline{L}_{ijk} = \mu L_{ijk} + \beta\gamma(\beta - \rho L^2)(L_i L_{jk} + L_j L_{ik} + L_k L_{ij}) - L\gamma(\beta - \rho L^2)(L_i L_{jk} + L_j L_{ik} + L_k L_{ij}) - \xi(L_i L_j b_k + L_i L_k b_j + L_j L_k b_i) + \varphi(L_i b_j b_k + L_j b_i b_k + L_k b_i b_j) + \eta L_i L_j L_k + \delta b_i b_j b_k$.

Since $\overline{L}_{ij|k} = 0\ in\ \overline{F}^n, after$ using (2.1), we have

(2.4) $$\partial_k\overline{L}_{ij} - \overline{L}_{ijr}\overline{G}^r_k - \overline{L}_{jr}\overline{F}^r_{ik} - \overline{L}_{ir}\overline{F}^r_{jk} = 0$$

Substituting in the above equation the values of $\partial_k\overline{L}_{ij},\ \overline{L}_{ir}\ and \overline{L}_{ijk}$ from (2.3) in (2.4) and then contracting the equation thus obtained with y^k, we get

(2.5) $$2\overline{L}_{ijr}D^r + \overline{L}_{jr}D^r_i + \overline{L}_{ir}D^r_j - L\gamma(Lb_j - \beta L\rho)(r_{i0} + s_{i0}) - L\gamma(Lb_i - \beta L_i)(r_{j0} + s_{j0}) - \{\gamma L(\rho L^2 - \beta)L_{ij} - \xi L_i L_j + \varphi(L_i b_j + L_j b_i) + \delta b_i b_j\}r_{00} - Lq\rho_0 L_{ij} - 2\rho q L_r L_{ij} G^r = 0 ,$$

where $'0'$ stands for contraction with y^k ,viz., $r_{j0} = r_{jk}y^k,\ r_{00} = r_{ij}y^i y^j,\ \rho_0 = \rho_k y^k$ and we have used the fact that $D^i_{jk}y^k = {}^cD^i_{jk}y^k = D^i_j$ [9].

Next, we deal with $\overline{L}_{i|j} = 0$,that is,

(2.6) $\partial_j\overline{L}_i - \overline{L}_{ir}\overline{G}^r_j - \overline{L}_r\overline{F}^r_{ij} = 0$.

Putting the values of $\partial_j\overline{L}_i, \overline{L}_{ir}\ and\ \overline{L}_r$ from (2.3) in (2.6) , we get ,

$$qb_{i|j} = [\mu L_{ir} + \gamma\{\beta^2 L_i L_r - L\beta(L_i b_r + L_r b_i) + L^2 b_i b_r\}]D^r_j + (pL_r + qb_r)\ {}^cD^r_{ij} + L\gamma(\beta L_i - Lb_i)(r_{0j} + s_{0j}) ,$$

which, after using $2r_{ij} = b_{i|j} + b_{j|i}$ *and* $2s_{ij} = b_{i|j} - b_{j|i}$,

gives

(2.7) $$2q\, r_{ij} = [\mu L_{ir} + \gamma\{\beta^2 L_i L_r - L\beta(L_i b_r + L_r b_i) + L^2 b_i b_r\}]D^r_j + [\mu L_{jr} + \gamma\{\beta^2 L_j L_r - L\beta(L_j b_r + L_r b_j) + L^2 b_j b_r\}]D^r_i + 2(pL_r + qb_r)\, {}^cD^r_{ij} + L\gamma(\beta L_i - Lb_i)(r_{0j} + s_{0j}) + L\gamma(\beta L_j - Lb_j)(r_{i0} + s_{i0}),$$

(2.8) $$2qs_{ij} = [\mu L_{ir} + \gamma\{\beta^2 L_i L_r - L\beta(L_i b_r + L_r b_i) + L^2 b_i b_r\}]D^r_j - [\mu L_{jr} + \gamma\{\beta^2 L_j L_r - L\beta(L_j b_r + L_r b_j) + L^2 b_j b_r\}]D^r_i + L\gamma(\beta L_i - Lb_i)(r_{0j} + s_{0j}) - L\gamma(\beta L_j - Lb_j)(r_{i0} + s_{i0}).$$

Subtracting (2.7) from (2.5) and contracting the resulting equation with y^i, we get

(2.9) $$-2[\mu L_{jr} + \gamma\{\beta^2 L_j L_r - L\beta(L_j b_r + L_r b_j) + L^2 b_j b_r\}]D^r + L\gamma(Lb_j - \beta L_j)r_{00} + 2qr_{0j} = 2\overline{L}_r D^r_j$$

Contracting (2.9) with y^j , we get

(2.10) $$(pL_r + qb_r)D^r = \frac{1}{2}\, q\, r_{00}$$

Subtracting (2.8) from (2.5) and contracting the resulting equation with y^i , we get

(2.11) $$[\mu L_{ir} + \beta\gamma L_i L_j - \beta L\gamma(L_i b_r + L_r b_i) + L^2\gamma b_i b_r]D^r = qs_{i0} + \frac{L\gamma}{2}(Lb_i - \beta L_i)r_{00}.$$

In view of $LL_{ir} = g_{ir} - L_i L_r$, equation (2.11) can be written as

(2.12) $$\frac{\mu}{L}\, g_{ir}D^r + \{(\beta^2\gamma - \frac{p}{L} - \rho q)L_i - \beta L\gamma b_i\}L_r D^r + L\gamma(Lb_i - \beta L_i)b_r D^r = qs_{i0} + \frac{1}{2}L\gamma(Lb_i - \beta L_i)r_{00}.$$

Contracting (2.12) by $b^i = g^{ij} b_j$, we get

(2.13) $$-2\beta(L^3\gamma\Delta + \mu)L_r D^r + 2L(L^3\gamma\Delta + \mu)b_r D^r = L^2(2qs_0 + L^2\gamma\Delta r_{00}) ,$$

where $\Delta = b^2 - \frac{\beta^2}{L^2}$

The equations (2.10) and (2.13) are algebraic equations in $L_r D^r$ *and* $b_r D^r$, whose solution is given by

(2.14) $$L_r D^r = \frac{Lq(\mu r_{00} - 2Lqs_0)}{2\overline{L}(L^3\Delta\gamma + \mu)}$$ and

(2.15) $$b_r D^r = \frac{2L^2 pqs_0 + \{L^3\gamma\overline{L}\Delta + \beta q\mu\}r_{00}}{\overline{L}(L^3\gamma\Delta + \mu)}$$

Contracting (2.12) by g^{ij} and putting the values of $L_r D^r$, and $b_r D^r$, we get

(2.16) $D^i = \frac{\mu r_{00} - 2Lqs_0}{2\mu(L^3\gamma\Delta+\mu)}\left[L^3\gamma b^i + \overline{L}^{-1}\{\mu q - \overline{L}\beta\gamma\}y^i\right] + \frac{Lq}{\mu}s_0^i,$

where $l^i = y^i L^{-1}$.

Proposition (2.1): The difference tensor $D^i = \overline{G}^i - G^i$ of generalized h-Randers change of Finsler metric is given by (2.16).

3. Projective Change of Finsler Metric

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$. It is well known that the change $L \to \overline{L}$ is projective if $\overline{G}^i = G^i + P(x,y)y^i$, where P(x,y) is a homogeneous scalar function of degree one in y^i, called projective factor [11].

Thus from (2.1) it follow that $L \to \overline{L}$ is projective iff $D^i = Py^i$. Now we consider that the generalized h-Randers change $L \to \overline{L} = (L^m + \beta^m)^{1/m}$ *is* projective.Then from equation (2.16), we have

(3.1) $Py^i = \frac{\mu r_{00} - 2Lqs_0}{2\mu(L^3\gamma\Delta+\mu)}\left[L^3\gamma b^i + \overline{L}^{-1}\{\mu q - \overline{L}\beta L\gamma\}y^i\right] + \frac{Lq}{\mu}s_0^i$.

Contracting (3.1) with $y_i(= g_{ij}y^j)$ and using the fact that $s_0^i y_i = 0$ and $y_i y^i = L^2$, we get

(3.2) $P = \frac{q\mu r_{00} - 2Lqs_0}{2\overline{L}(L^3\gamma\Delta+\mu)}$.

Putting the value of P from (3.2) in (3.1), we get

(3.3) $\frac{L\gamma\{\mu r_{00} - 2Lqs_0\}}{2\mu(L^3\gamma\Delta+\mu)}\left(\beta y^i - L^2 b^i\right) = \frac{Lq}{\mu}s_0^i$,

Transvecting (3.3) by b_i, we get

(3.4) $r_{00} = \frac{-2qs_0}{L^2\gamma\Delta}$.

Substituting the value of r_{00} from (3.4) in (3.2), we get

(3.5) $P = \frac{-q^2 s_0}{\overline{L}L^2\gamma\Delta}$.

Substituting the value of r_{00} from (3.4) in (3.3), we get

(3.6) $s_0^i = \left(b^i - \frac{\beta}{L^2}y^i\right)\frac{s_0}{\Delta}$.

The equations (3.4) and (3.6) give the necessary condition under which a generalized h-Randers change becomes a projective change.

Conversely, if condition (3.4) and (3.6) are satisfied, then putting these conditions in (2.16), we get

$D^i = \frac{-q^2 s_0}{L^2 \bar{L} \gamma \Delta} y^i$,   i.e. $D^i = P y^i$.

Thus $\overline{F}^n$ is projective to F^n .

Theorem 3.1: The generalized h –Randers change of Finsler metric is projective iff (3.4) and (3.6) hold good; the projective factor P is given by (3.5).

4. Douglas Space

The Finsler space F^n is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [12] . We shall write hp(r) to denote a homogeneous polynomial in y^i of degree r .If we write

$B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

(4.1) $$B^{ij} = \frac{(\mu r_{00} - 2Lqs_0)L^3\gamma}{2\mu(L^3\gamma\Delta + \mu)}\left(b^i y^j - b^j y^i\right) + \frac{Lq}{\mu}\left(s_0^i y^j - s_0^j y^i\right) .$$

If a Douglas space is transformed to a Douglas space by a generalized h-Randers change (2.1) then B^{ij} must be hp (3) and vice-versa.

Theorem: The generalized h-Randers change of Douglas space is a Douglas space iff B^{ij} given by (4.1) is hp (3).

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