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Article

Characterization and Bayesian Estimation of Minimax Distribution

Kawsar Fatima* and S.P Ahmad

Department of Statistics, University of Kashmir, Srinagar, India

*Author to whom correspondence should be addressed; E-Mail: Kawsarfatima@gmail.com

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Abstract: The main objective of our research problem is to study the Bayesian Analysis of Minimax distribution under single & double priors. Simulation study will be performed to compare the performance of the posterior estimates under various priors in R Software.

Keywords: Bayesian Analysis, Minimax distribution, priors, Maximun Likelihood Estimation, Priors & R Software.

Mathematics Subject Classification (2010):68M15, 62F15

1. Introduction

There are various types of life time models such as exponential, Weibull and Gamma that are used in reliability and life testing. Although many alternatives and generalizations, it is fair to say that the Beta distribution of first kind provides the premier family of continuous distributions on bounded support. The probability density function of generalized beta distribution is given by

$$f(x;\alpha,\delta,\theta)=\frac{\alpha x^{\alpha\delta-1}(1-x^{\alpha})^{\theta-1}}{B(\delta,\theta)}; \qquad 0<x<1 \qquad (1.1)$$

If we put $\alpha=1$, the equation (1.1) reduces to beta distribution of first kind with probability density function as:

$$f(x;\delta,\theta)=\frac{1}{B(\delta,\theta)}\,x^{\delta-1}(1-x)^{\theta-1};\qquad 0<x<1 \tag{1.2}$$

where δ and θ are positive real quantities and the variable X satisfies $0\le x\le 1$. The quantity $B(\delta,\theta)$ is the beta function. Equation (1.2) is also known as the standard beta or classical beta distribution.

If we put $\delta=1$, the equation (1.1) reduces to minimax distribution with probability density function as:

$$f(x;\alpha,\theta)=\frac{1}{B(1,\theta)}[\alpha\, x^{\alpha-1}(1-x^{\alpha})]^{\theta-1};\qquad 0<x<1$$

$$\Rightarrow f(x;\theta,\alpha)=\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\quad ;0<x<1\ ,\quad \theta,\alpha>0 \tag{1.3}$$

The cumulative distribution function (CDF) of Minimax distribution is:

$$F(x;\theta,\alpha)=1-\ (1-x^{\alpha})^{\theta}\quad ;0<x<1\ ,\quad \theta,\alpha>0 \tag{1.4}$$

Here α and θ are two shape parameters. Minimax distribution is a special case of generalized beta distribution, which was proposed by McDonald (1984). Jones (2007) explored the genesis of the Minimax distribution and made some similarities between beta and Minimax distribution. The probability density function of Minimax distribution are unimodal, uniantimodal, increasing, decreasing or constant depending on the values of α and θ. Both beta and Minimax densities are log-concave if and only if both their parameters are greater than or equal to 1. It can be showed that the Minimax distribution has the same basic shape properties as the beta distribution, namely:

$\alpha>1,\theta>1\Rightarrow$unimodal; $\alpha<1,\theta<1\Rightarrow$uniantimodal; $\alpha>1,\theta\le 1\Rightarrow$increasing;

$\alpha\le 1,\theta>1\Rightarrow$decreasing; $\alpha=1,\theta=1\Rightarrow$constant

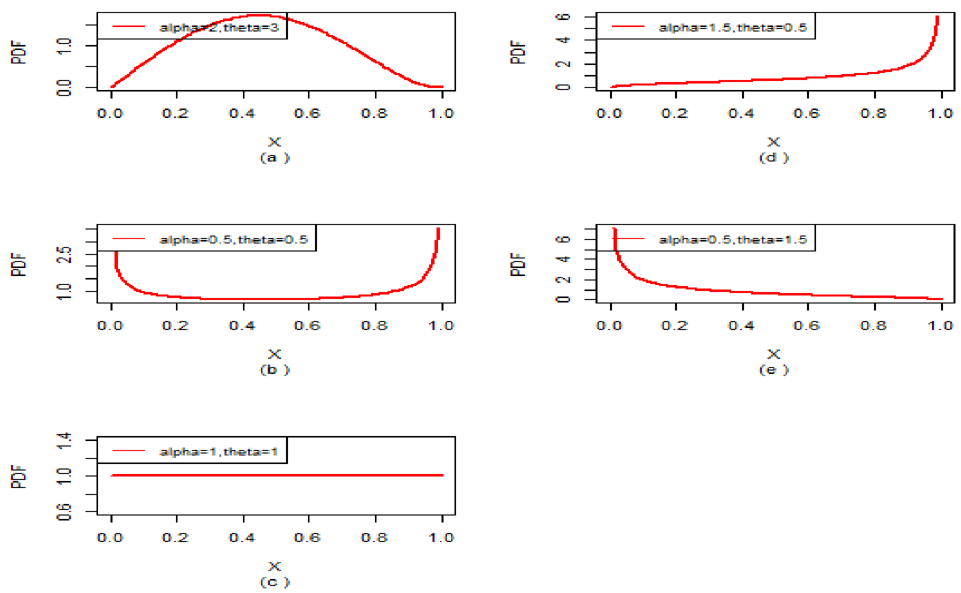

Figure 1. Minimax densities for selected values of (α andθ) ;(a) α=2 and θ=3; (b) α=0.5 and θ=0.5; (c) α=1 and θ=1; (d) α=1.5 and θ=0.5; (e) α=0.5 and θ=1.5;

Shadrokh and Pazira (2010) studied the minimax estimation of the Minimax distribution; Makhdoom (2011) obtain the maximum likelihood and moment estimators of the parameters of the minimax distribution in the presence of one outlier while Deiri (2011) considered the Estimation of the parameters of minimax distribution with presence of two outliers. R. Sultan et.al (2014) discussed Bayesian analysis of the power function distribution under double priors. Lanping Li (2015) obtained the Bayes estimates of the shape parameter of Minimax distribution under different loss functions especially entropy loss, LINEX loss, precautionary loss, and squared error loss function.

2. Special Cases

Case 1: When $\alpha = \theta = 1$, then Minimax distribution (1.3) reduces to Uniform distribution (UD) with probability density function as:

$$f(x) = 1; \quad 0 < x < 1 \tag{2.1}$$

Case 2: When $\theta = 1$, then Minimax distribution (1.3) reduces to Power distribution (PD) with probability density function as:

$$f(x;\alpha) = \alpha\, x^{\alpha-1}; \quad 0 < x < 1 \quad ,\alpha > 0 \tag{2.2}$$

Case 3: When $\alpha = 1$, then Minimax distribution (1.3) reduces to one parameter Minimax distribution with probability density function as:

$$f(x;\theta) = \theta\,(1-x)^{\theta-1}; \quad 0 < x < 1 \quad ,\theta > 0 \tag{2.3}$$

3. Reliability Analysis

(i) Reliability function R(x)

The reliability function or survival function $R(x)$ .This function can be derived using the cumulative distribution function and is given by

$$R(x) = (1 - F(x))$$

$$R(x) = (1 - x^{\alpha})^{\theta} \quad ; 0 < x < 1 \tag{3.1}$$

(ii) Hazard Function H(x)

The hazard or instantaneous rate function is denoted by $H(x)$. The hazard function of x can be interpreted as instantaneous rate or the conditional probability density of failure at time x, given that the unit has survived until x. The hazard function is defined to be

$$H(x) = \frac{f(x)}{R(x)} = \frac{\alpha\theta x^{\alpha-1}}{(1-x^{\alpha})} \quad ; \quad 0 < x < 1 \tag{3.2}$$

(iii) Reverse Hazard function $\phi(x)$

The reverse hazard function can be interpreted as an approximate probability of failure in $[x, x+d]$, given that the failure had occurred in $[0, x]$. The reverse hazard function $\phi(x)$ is defined to be

$$\phi(x) = \frac{f(x)}{F(x)} = \frac{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}}{1-(1-x^{\alpha})^{\theta}} \tag{3.3}$$

4. Structural Properties of Minimax Distribution

4.1. Theorem

Let $X = (x_1, x_2, \ldots, x_n)$ be a random sample of size n from the Minimax distribution with probability density function

$$f(x;\theta,\alpha) = \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1} \quad ; 0 < x < 1 , \quad \theta,\alpha > 0$$

Then $$E(X^r) = \theta B\left(\frac{r}{\alpha}+1,\theta\right) = \frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{r}{\alpha}+1+\theta\right)} \qquad r = 1, 2, \ldots$$

Proof: Since we know that the r^th moment of a random variable x is given by

$$E(X^r) = \int_0^1 X^r f(x;\theta,\alpha)\, dx \tag{4.1}$$

Now using eq. (1.3) in eq. (4.1), we have

$$E(X^r) = \int_0^1 X^r \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\, dx$$

On solving the above equation, we get

$$E(X^r) = \theta B\left(\frac{r}{\alpha}+1,\theta\right) = \frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{r}{\alpha}+1+\theta\right)} \tag{4.2}$$

4.2. Moments of Minimax Distribution

If we put r=1 in eq. (4.2), we get the mean of Minimax distribution which is given by

$$\mu_1' = \frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)} \tag{4.3}$$

If we put r=2 in eq. (4.2), we have

$$\mu_2' = \frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)} \tag{4.4}$$

Thus the variance of Minimax distribution is given by

$$\mu_2 = \frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)} - \left\{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2 \tag{4.5}$$

If we put r=3 in eq. (4.2), we have

$$\mu_3' = \frac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)} \tag{4.6}$$

Thus $$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \tag{4.7}$$

After substituting the values of eq. (4.3), eq. (4.4) and eq. (4.6) in eq. (4.7), we have

$$\mu_3 = \frac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)} - 3\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}\ \frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)} + 2\left\{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2 \tag{4.8}$$

If we put r=4 in eq. (4.2), we have

$$\mu_4' = \frac{\Gamma\left(\frac{4}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{4}{\alpha}+1+\theta\right)} \tag{4.9}$$

Thus, $$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 \tag{4.10}$$

After substituting the values of eq. (4.3), eq. (4.4), eq. (4.6) and eq. (4.9) in eq. (4.10), we have

$$\mu_4 = \frac{\Gamma\left(\frac{4}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{4}{\alpha}+1+\theta\right)} - 4\frac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)}\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)} + 6\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)} \times\left\{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2 - 3\left\{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^4 \tag{4.11}$$

4.3. Coefficient of Variation

It is the ratio of standard deviation and mean. Usually it is denoted by C.V. and is given by

$$C.V = \frac{\sigma}{\mu} \tag{4.12}$$

By using the value of eq. (4.3) and eq. (4.5) in eq. (4.12), we get

$$C.V = \frac{\sqrt{\left[\frac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)} - \left\{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2\right]}}{\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}} \tag{4.13}$$

4.4. Skewness and Kurtosis of Minimax Distribution

(i) Skewness: The most popular way to measure the Skewness and kurtosis of a distribution function rests upon ratios of moments. Lack of symmetry of tails (about mean) of frequency distribution curve is known as Skewness. The formula for measure of Skewness given by Karl Pearson in terms of moments of frequency distribution is given by

$$\beta_1 = \frac{\mu_3^{\ 2}}{\mu_2^{\ 3}} \tag{4.14}$$

After using eq. (4.5) and eq. (4.8) in eq. (4.14), we have

$$\beta_1=\frac{\left[\dfrac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)}-3\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}+2\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2\right]^2}{\left[\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}-\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2\right]^3}$$

$$\gamma_1=\sqrt{\beta_1}=\frac{\dfrac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)}-3\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}+2\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2}{\left[\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}-\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2\right]^{3/2}} \qquad (4.15)$$

(ii) Kurtosis: Kurtosis is the degree of peakedness of a distribution, defined as normalized form of the fourth central moment μ_4 of a distribution. There are several flavors of kurtosis commonly encountered, including the kurtosis proper, denoted β_2 and defined by

$$\beta_2=\frac{\mu_4}{\mu_2^2} \qquad (4.16)$$

After using eq. (4.11) and eq. (4.5) in eq. (4.16), we have

$$\beta_2=\frac{\left\{\begin{array}{l}\dfrac{\Gamma\left(\frac{4}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{4}{\alpha}+1+\theta\right)}-4\dfrac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)}\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\\+6\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2-3\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^4\end{array}\right\}}{\left[\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}-\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2\right]^2} \qquad (4.17)$$

$$\gamma_2=\beta_2-3$$

Using equation (4.17)

$$\gamma_2 = \frac{\left\{\begin{array}{l} \dfrac{\Gamma\left(\frac{4}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{4}{\alpha}+1+\theta\right)} - 4\dfrac{\Gamma\left(\frac{3}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{3}{\alpha}+1+\theta\right)}\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)} \\ +6\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)}\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2 - 3\left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^4 \end{array}\right\}}{\left[\dfrac{\Gamma\left(\frac{2}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{2}{\alpha}+1+\theta\right)} - \left\{\dfrac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{1}{\alpha}+1+\theta\right)}\right\}^2\right]^2} - 3 \quad (4.18)$$

5. Mode and Median of Minimax Distribution

5.1. Mode of Minimax Distribution

The mode of Minimax distribution occurs at

$$\frac{d\, f(x;\theta,\alpha)}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\right) = 0$$

$$\Rightarrow [(\alpha-1)-(\theta-1)(1-x^{\alpha})^{-1}\alpha x^{\alpha}] = 0$$

$$\Rightarrow (\alpha-1) = (\theta-1)(1-x^{\alpha})^{-1}\alpha x^{\alpha}$$

$$\Rightarrow (\alpha-1)(1-x^{\alpha}) = (\theta-1)\alpha x^{\alpha}$$

$$\Rightarrow (\alpha-1)(1-x^{\alpha}) = (\theta-1)\alpha x^{\alpha}$$

$$\Rightarrow \qquad x^{\alpha} = \frac{1-\alpha}{1-\theta\alpha}$$

$$\Rightarrow \qquad x = \left(\frac{1-\alpha}{1-\theta\alpha}\right)^{1/\alpha} \quad for\ \alpha \ge 1, \quad \theta \ge 1, \quad (\alpha,\theta) \ne (1,1) \quad (5.1)$$

5.2 .Median of Minimax Distribution

Let X be a continuous random variable, the median of X is the number M such that

$$P(X \le M) = \frac{1}{2}$$

Then $P(M \le X) = \frac{1}{2}$ also.

If f is the probability density function for X and f is defined as (0, 1), then we can calculate M by solving the equation

$$P(0 \le X \le M) = \int_0^M f(x)dx = \frac{1}{2} \text{ for } M.$$

$$\Rightarrow P(0 \le X \le M) = \int_0^M \alpha\theta x^{\alpha-1}(1-x^\alpha)^{\theta-1} dx = \frac{1}{2}$$

$put \quad x^\alpha = y \qquad ;as \;\; x \to 0, y \to 0$

$\alpha x^{\alpha-1} dx = dy \qquad ;as \;\; x \to M, y \to M^\alpha$

$$\Rightarrow \qquad \theta \int_0^{M^\alpha} (1-y)^{\theta-1} dy = \frac{1}{2}$$

$$\Rightarrow \qquad -\left[(1-y)^\theta\right]_0^{M^\alpha} = \frac{1}{2}$$

$$\Rightarrow \qquad -(1-M^\alpha)^\theta + 1 = \frac{1}{2}$$

Applying log on both sides, we have

$$\theta \log(1-M^\alpha) = \log\left(\frac{1}{2}\right)$$

$$\Rightarrow \qquad \log(1-M^\alpha) = -\frac{1}{\theta}\log 2$$

$$\Rightarrow \qquad (1-M^\alpha) = e^{\log 2^{-1/\theta}}$$

$$\Rightarrow \qquad M = \left(1-2^{-1/\theta}\right)^{1/\alpha} \tag{5.2}$$

6. Moment Generating Function and Characteristic Function

Theorem 6.1. Let X have a Minimax distribution. Then moment generating function of X denoted by $M_X(t)$ is given by:

$$M_X(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \left\{ \frac{\Gamma\left(\frac{i}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{i}{\alpha}+1+\theta\right)} \right\} \tag{6.1}$$

Proof: By definition

$$M_X(t) = E(e^{tx}) = \int_0^1 e^{tx} f(x;\alpha,\theta)dx$$

Using Taylor series

$$M_X(t) = \int_0^1 \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f(x;\alpha,\theta)dx$$

$$\Rightarrow \qquad M_X(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \int_0^1 x^i f(x;\alpha,\theta)dx$$

$$\Rightarrow \qquad M_X(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i)$$

$$\Rightarrow \qquad M_X(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \left\{ \frac{\Gamma\left(\frac{i}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{i}{\alpha}+1+\theta\right)} \right\}$$

This completes the proof.

Theorem 6.2. Let X have a Minimax distribution. Then characteristic function of X denoted by $\phi_X(t)$ is given by:

$$\phi_X(t) = \sum_{i=0}^{\infty} \frac{(it)^i}{i!} \left\{ \frac{\Gamma\left(\frac{i}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{i}{\alpha}+1+\theta\right)} \right\} \qquad (6.2)$$

Proof: By definition

$$\phi_X(t) = E(e^{itx}) = \int_0^1 e^{itx} f(x;\alpha,\theta)dx$$

Using Taylor series

$$\phi_X(t) = \int_0^1 \left(1 + itx + \frac{(itx)^2}{2!} + \cdots\right) f(x;\alpha,\theta)dx$$

$$\Rightarrow \qquad \phi_X(t = \sum_{i=0}^{\infty} \frac{(it)^i}{i!} \int_0^1 x^i f(x;\alpha,\theta)dx$$

$$\Rightarrow \qquad \phi_X(t) = \sum_{i=0}^{\infty} \frac{(it)^i}{i!} E(X^i)$$

$$\Rightarrow \qquad \phi_X(t) = \sum_{i=0}^{\infty} \frac{(it)^i}{i!} \left\{ \frac{\Gamma\left(\frac{i}{\alpha}+1\right)\Gamma(\theta+1)}{\Gamma\left(\frac{i}{\alpha}+1+\theta\right)} \right\}$$

This completes the proof.

7. Harmonic Mean of Minimax Distribution

The harmonic mean (H) is given as:

$$\frac{1}{H} = E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x}\, f(x;\alpha,\theta)dx$$

$$\frac{1}{H} = \int_0^1 \frac{1}{x}\, \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\, dx$$

$$\frac{1}{H} = \theta B\left(1-\frac{1}{\alpha},\theta\right)$$

$$\frac{1}{H} = \frac{\Gamma(1-1/\alpha)\Gamma(\theta+1)}{\Gamma(1-1/\alpha+\theta)} \qquad ;where\, \Gamma(\theta+1) = \theta\Gamma\theta \qquad (7.1)$$

8. Order Statistics and Entropy of Minimax Distribution

8.1. Order Statistics

Let $X_{(1)}$ denote the smallest of $\{X_1, X_2, \ldots, X_n\}$, $X_{(2)}$ denote the second smallest of $\{X_1, X_2, \ldots, X_n\}$, and similarly $X_{(k)}$ denote the kth smallest of $\{X_1, X_2, \ldots, X_n\}$. Then the random variables $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, called the order statistics of the sample $\{X_1, X_2, \ldots, X_n\}$, has probability density function of the k^{th} order statistic, $X_{(k)}$, as:

$$f_X(k) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k} \qquad (8.1)$$

for $k = 1,2,\ldots,n$.

The pdf of the k^{th} order statistic is defined as:

$$f_X(k) = \frac{n!}{(k-1)!(n-k)!} \alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}[1-(1-x^{\alpha})^{\theta}]^{k-1}[(1-x^{\alpha})^{\theta}]^{n-k} \qquad (8.2)$$

for $k = n$ in (8.2)

The pdf of the largest statistics $X_{(n)}$ is therefore:

$$f_X(n) = n\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}[1-(1-x^{\alpha})^{\theta}]^{n-1} \tag{8.3}$$

for $k=1$ in (8.2)

and the pdf of the smallest order statistic $X_{(1)}$ is given by:

$$f_X(1) = n\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}[(1-x^{\alpha})^{\theta}]^{n-1} \tag{8.4}$$

8.2. Entropy of Minimax Distribution

$$H(x) = -E[\log f(x)] = -E\{\log\{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\}]$$

$$= -\log[\alpha\theta] - (\alpha-1)E(\log x) - (\theta-1)E(\log(1-x^{\alpha})) \tag{8.5}$$

$$Now, E(\log x) = \int_0^1 \log x\{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\}dx = \theta\int_0^1 \alpha\log x\{x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\}dx$$

$$= \theta\int_0^1 \log x^{\alpha}\{x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\}dx \tag{8.6}$$

$$put \quad x^{\alpha} = y \qquad ; as \quad x \to 0, y \to 0$$

$$x^{\alpha-1}dx = \frac{dy}{\alpha} \qquad ; as \quad x \to 1, y \to 1$$

$$= \frac{\theta}{\alpha}\int_0^1 \log y\{y^{1-1}(1-y)^{\theta-1}\}dy$$

$$= \frac{\theta}{\alpha}\frac{\Gamma 1\Gamma\theta}{\Gamma(\theta+1)}[\psi(1)-\psi(\theta+1)]$$

$$\Rightarrow E(\log x) = \frac{1}{\alpha}[\psi(1)-\psi(\theta+1)] \tag{8.7}$$

$$E(\log(1-x^{\alpha})) = \int_0^1 \log(1-x^{\alpha})\{\alpha\theta x^{\alpha-1}(1-x^{\alpha})^{\theta-1}\}dx \tag{8.8}$$

$$put \quad x^{\alpha} = y \qquad ; as \quad x \to 0, y \to 0$$

$$x^{\alpha-1}dx = \frac{dy}{\alpha} \qquad ; as \quad x \to 1, y \to 1$$

$$= \frac{\theta\alpha}{\alpha}\int_0^1 \log(1-y)\{y^{1-1}(1-y)^{\theta-1}\}dy$$

$$= \frac{\theta\alpha}{\alpha}\frac{\Gamma 1\Gamma\theta}{\Gamma(\theta+1)}[\psi(\theta)-\psi(\theta+1)]$$

$$\Rightarrow E(\log(1-x^{\alpha})) = [\psi(\theta)-\psi(\theta+1)] \tag{8.9}$$

Substitute the values of (8.7) and (8.9) in equation (8.5)

$$H(x) = \frac{(\alpha-1)}{\alpha}[\psi(\theta+1)-\psi(1)]+(\theta-1)[\psi(\theta+1)-\psi(\theta)]-\log[\alpha\theta] \tag{8.10}$$

9. Maximum Likelihood Estimation for the Shape Parameter θ of Minimax Distribution Assuming Shape Parameter α Is to Be Known

Let us consider a random sample $\underline{x}=(x_1,x_2,...,x_n)$ of size n from the Minimax Distribution. Then the likelihood function for the given sample observation is

$$L(\underline{x}\,|\,\theta) = \alpha^n\theta^n\prod_{i=1}^{n} x^{\alpha-1}\prod_{i=1}^{n}(1-x^{\alpha})^{\theta-1} \tag{9.1}$$

The log-likelihood function is

$$\ln L(\underline{x}\,|\,\theta) = n\ln\alpha + n\ln\theta + \sum_{i=1}^{n}\ln(x_i{}^{\alpha-1}) + (\theta-1)\sum_{i=1}^{n}\ln(1-x_i{}^{\alpha}) \tag{9.2}$$

As shape parameter α is assumed to be known, the ML estimator of shape parameter θ is obtained by solving the

$$\frac{\partial \ln L(\underline{x}\,|\,\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln(1-x_i{}^{\alpha}) = 0$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln(1-x_i{}^{\alpha})} \tag{9.3}$$

10. Posterior Distribution of Unknown Parameter θof Minimax Distribution under Double and Single Priors

10.1.Selection of Double Priors for the Unknown Parameter θ of Minimax Distribution Assuming Shape Parameter α is to be Known

10.1.1. Posterior distribution of unknown parameter θ of minimax distribution under Gamma-exponential distribution as double prior

Assume that the prior distribution of θ is gamma distribution with hyper parameters a_1 $\& b_1$ given below:

$$g_1(\theta) = \frac{b_1{}^{a_1}}{\Gamma a_1}e^{-\theta b_1}\theta^{a_1-1}\ ; \qquad 0<\theta<\infty;\ a_1,b_1>0 \tag{10.1}$$

Secondly assume that the prior distribution of θ to be exponential with hyper parameter c_1.

$$g_2(\theta) = c_1 e^{-\theta c_1}\ , \quad 0 < \theta < \infty;\ c_1 > 0 \tag{10.2}$$

Now the double prior is defined as

$$\begin{aligned} g_{11}(\theta) &\propto g_1(\theta)\ g_2(\theta) \\ g_{11}(\theta) &\propto \theta^{a_1-1}\, e^{-(b_1+c_1)\theta} \end{aligned} \tag{10.3}$$

Thus posterior distribution of θ forgiven data X is

$$\pi_1(\theta\,|\,\underline{x}) \propto L(\underline{x}\,|\,\theta)\, g_{11}(\theta)$$

$$\pi_1(\theta\,|\,\underline{x}) \propto \alpha^n \theta^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x_i^{\alpha})^{\theta-1}\, \theta^{a_1-1}\, e^{-(b_1+c_1)\theta}$$

$$\begin{aligned} \pi_1(\theta\,|\,\underline{x}) &= K\theta^{n+a_1-1}\, e^{-\theta\left(b_1+c_1-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)} \\ \pi_1(\theta\,|\,\underline{x}) &= K\theta^{n+a_1-1}\, e^{-\theta\beta_1} \end{aligned} \tag{10.4}$$

where K is independent of θ, $\beta_1 = \left(b_1 + c_1 - \sum_{i=1}^{n} \ln(1-x_i^{\alpha})\right)$

and $K^{-1} = \int_0^{\infty} \theta^{n+a_1-1}\, e^{-\theta\beta_1}\, d\theta$

$$\Rightarrow K^{-1} = \frac{\Gamma(n+a_1)}{\beta_1^{\,n+a_1}}$$

Therefore from (10.4) we have

$$\pi_1(\theta\,|\,\underline{x}) = \frac{\beta_1^{\,n+a_1}}{\Gamma(n+a_1)}\ \theta^{n+a_1-1}\, e^{-\theta\beta_1}\quad ;\quad \theta > 0 \tag{10.5}$$

which is the pdf of gamma distribution with parameters $\gamma_1 = (n+a_1)$ & $\beta_1 = \left(b_1 + c_1 - \sum_{i=1}^{n} \ln(1-x_i^{\alpha})\right)$

10.1.2. Posterior distribution under Chi-square-Exponential as a prior

Now assuming that the prior distribution of θ is chi-square distribution with hyper parameters a_2.

$$g_3(\theta) = \frac{1}{\Gamma(a_2/2)2^{a_2/2}}\theta^{\frac{a_2}{2}-1}\, e^{-\frac{\theta}{2}}\ ;\quad 0 < \theta < \infty\ , a_2 > 0 \tag{10.6}$$

Another prior be exponential distribution with hyper parameter c_2.

$$g_4(\theta) = \frac{1}{c_2} e^{\frac{-\theta}{c_2}}\ , \quad 0 < \theta < \infty;\ c_2 > 0 \tag{10.7}$$

Now the double prior is defined as

$$g_{22}(\theta) \propto \theta^{\frac{a_2}{2}-1} e^{-\left(\frac{1}{c_2}+\frac{1}{2}\right)\theta} \tag{10.8}$$

Thus posterior distribution of θ is given as

$$\pi_2(\theta \mid \underline{x}) \propto \alpha^n \theta^n \prod_{i=1}^{n} x_i^{\alpha-1} \prod_{i=1}^{n} (1-x_i^{\alpha})^{\theta-1} \theta^{\frac{a_2}{2}-1} e^{-\left(\frac{1}{c_2}+\frac{1}{2}\right)\theta}$$

$$= K\theta^{n+\frac{a_2}{2}-1} e^{-\theta\left(\frac{1}{c_2}+\frac{1}{2}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)}$$

$$= K\theta^{n+\frac{a_2}{2}-1} e^{-\theta\beta_2} \tag{10.9}$$

Where K is independent of θ, $\beta_2 = \left(\frac{1}{c_2}+\frac{1}{2}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)$ and

$$K^{-1} = \int_0^{\infty} \theta^{n+\frac{a_2}{2}-1} e^{-\theta\beta_2}\, d\theta$$

$$\Rightarrow K^{-1} = \frac{\Gamma\left(n+\frac{a_2}{2}\right)}{\beta_2^{\,n+\frac{a_2}{2}}}$$

Therefore from (10.9) we have

$$\pi_2(\theta \mid \underline{x}) = \frac{\beta_2^{\,n+\frac{a_2}{2}}}{\Gamma\left(n+\frac{a_2}{2}\right)} \theta^{n+\frac{a_2}{2}-1} e^{-\theta\beta_2} \qquad \theta > 0 \tag{10.10}$$

Which is the pdf of gamma distribution with parameters $\gamma_2 = (n+\frac{a_2}{2})$ & $\beta_2 = \left(\frac{1}{c_2}+\frac{1}{2}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)$

10.1.3. Posterior distribution under Gamma-Chi-square as prior

Assuming double prior distribution of θ be gamma with hyper parameters a_3 & b_2 and chi-square distribution with hyper parameter a_4 .

$$g_{33}(\theta) \propto \theta^{a_3+\frac{a_4}{2}-2} e^{-\left(b_2+\frac{1}{2}\right)\theta} \tag{10.11}$$

Thus, posterior distribution of θ is given as

$$\pi_3(\theta \mid \underline{x}) = K\theta^{\left(n+a_3+\frac{a_4}{2}-1\right)-1} e^{-\theta\left(b_2+\frac{1}{2}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)}$$

$$\pi_3(\theta \mid \underline{x}) = K\theta^{\left(n+a_3+\frac{a_4}{2}-1\right)-1} e^{-\theta\beta_3} \tag{10.12}$$

Where K is independent of θ, $\beta_3 = \left(b_2 + \frac{1}{2} - \sum_{i=1}^{n} \ln(1-x_i{}^{\alpha})\right)$

and $K^{-1} = \int_0^{\infty} \theta^{\left(n+a_3+\frac{a_4}{2}-1\right)-1} e^{-\theta\beta_3}\, d\theta$

$$\Rightarrow K^{-1} = \frac{\Gamma\left(n+a_3+\frac{a_4}{2}-1\right)}{\beta_3{}^{n+a_3+\frac{a_4}{2}-1}}$$

Therefore from (10.12) we have

$$\pi_3(\theta \mid \underline{x}) = \frac{\beta_3{}^{n+a_3+\frac{a_4}{2}-1}}{\Gamma\left(n+a_3+\frac{a_4}{2}-1\right)}\ \theta^{\left(n+a_3+\frac{a_4}{2}-1\right)-1} e^{-\theta\beta_3} \quad \theta > 0 \tag{10.13}$$

which is the pdf of gamma distribution with parameters.

$\gamma_3 = (n + a_3 + \frac{a_4}{2} - 1)$ & $\beta_3 = \left(b_2 + \frac{1}{2} - \sum_{i=1}^{n} \ln(1-x_i{}^{\alpha})\right)$

Thus, Bayes estimators for θ under gamma-exponential Prior, chi-square-exponential prior and gamma-chi-square prior are

$$\hat{\theta}_{GEP} = \frac{n+a_1}{\left(b_1 + c_1 - \sum_{i=1}^{n} \ln(1-x_i{}^{\alpha})\right)} \ ;\hat{\theta}_{CEP} = \frac{\left(n+\frac{a_2}{2}\right)}{\left(\frac{1}{c_2} + \frac{1}{2} - \sum_{i=1}^{n} \ln(1-x_i{}^{\alpha})\right)} \ ;\hat{\theta}_{GCP} = \frac{\left(n+a_3+\frac{a_4}{2}-1\right)}{\left(b_2 + \frac{1}{2} - \sum_{i=1}^{n} \ln(1-x_i{}^{\alpha})\right)}$$

10.2. Posterior Predictive Distribution under double Priors

10.2.1. Posterior predictive distribution under Gamma-Exponential prior

The posterior predictive distribution for $Y_{n+1} = y_{n+1}\ given Y = y_1, y_2, \ldots, y_n$ under Gamma-Exponential prior is

$$\pi_{11}(y_{n+1} \mid Y) = \int_0^{\infty} \pi(y/\theta)\, \pi_1(\theta/x)\, d\theta$$

$$\Rightarrow \pi_{11}(y_{n+1} \mid Y) = \frac{\alpha\beta_1^{\gamma_1}}{\Gamma\gamma_1} y^{\alpha-1} \int_0^{\infty} \theta^{(\gamma_1+1)-1} (1-y^{\alpha})^{\theta-1} e^{-\beta_1\theta} d\theta$$

$$\Rightarrow \pi_{11}(y_{n+1} \mid Y) \ = \frac{\alpha\gamma_1\beta_1^{\gamma_1} y^{\alpha-1}}{(1-y^{\alpha})\left(\beta_1 - \ln(1-y^{\alpha})\right)^{\gamma_1+1}} \tag{10.14}$$

where $\gamma_1 = (n + a_1)$ & $\beta_1 = \left(b_1 + c_1 - \sum_{i=1}^{n} \ln(1 - x_i^{\alpha}) \right)$

10.2.2. Posterior predictive distribution under Chi-Square-Exponential Prior

The posterior predictive distribution for $Y_{n+1} = y_{n+1}$ given $Y = y_1, y_2, \ldots, y_n$ under Chi-Square-Exponential prior is

$$\pi_{22}(y_{n+1} \mid Y) = \frac{\alpha \beta_2^{\gamma_2}}{\Gamma \gamma_2} y^{\alpha-1} \int_0^{\infty} \theta^{(\gamma_2+1)-1} (1 - y^{\alpha})^{\theta-1} e^{-\beta_2 \theta} d\theta$$

$$\pi_{22}(y_{n+1} \mid Y) = \frac{\alpha \gamma_2 \beta_2^{\gamma_2} y^{\alpha-1}}{(1 - y^{\alpha}) \left(\beta_2 - \ln(1 - y^{\alpha}) \right)^{\gamma_2+1}} \tag{10.15}$$

where $\gamma_2 = (n + \frac{a_2}{2})$ & $\beta_2 = \left(\frac{1}{c_1} + \frac{1}{2} - \sum_{i=1}^{n} \ln(1 - x_i^{\alpha}) \right)$

10.2.3. Posterior predictive distribution under Gamma Chi-Square prior

The posterior predictive distribution for $Y_{n+1} = y_{n+1}\ given Y = y_1, y_2, \ldots, y_n$ under Gamma Chi-Square prior is

$$\pi_{33}(y_{n+1} \mid Y) = \frac{\alpha \beta_3^{\gamma_3}}{\Gamma \gamma_3} y^{\alpha-1} \int_0^{\infty} \theta^{(\gamma_3+1)-1} (1 - y^{\alpha})^{\theta-1} e^{-\beta_3 \theta} d\theta$$

$$\Rightarrow \pi_{33}(y_{n+1} \mid Y) = \frac{\alpha \gamma_3 \beta_3^{\gamma_3} y^{\alpha-1}}{(1 - y^{\alpha}) \left(\beta_3 - \ln(1 - y^{\alpha}) \right)^{\gamma_3+1}} \tag{10.16}$$

where $\gamma_3 = (n + a_3 + \frac{a_4}{2} - 1)$ & $\beta_3 = \left(b_2 + \frac{1}{2} - \sum_{i=1}^{n} \ln(1 - x_i^{\alpha}) \right)$

10.3. Comparison of Double Priors with Respect to Posterior Variances

The variances of the posterior distribution under all of assumed informative priors are calculated by assuming different set of values for hyper parameters, different sample size and different value of parameter which is given by

$$V(\theta / X) = \frac{\gamma_i}{\beta_i^{\ 2}}; i = 1,2,3. \tag{10.17}$$

where γ_i & β_i are shape and rate parameters of gamma distribution.

10.4. Selection of Single Priors for the Unknown Parameter θ of Minimax Distribution Assuming Shape Parameter α is to be known

10.4.1. Posterior distribution for the parameter of Minimax distribution under Erlang distribution as single prior

The pdf of Erlang distribution with hyper parameters a_4 & b_4 is given as

$$g(\theta)=\frac{1}{\Gamma(b_4)a_4^{\,b_4}}e^{-\frac{\theta}{a_4}}\,\theta^{b_4-1}\ , \qquad 0<\theta<\infty;\ a_4,b_4>0 \quad (10.18)$$

Thus posterior distribution of θ is given as

$$\pi(\theta\,|\,\underline{x})\propto\alpha^n\theta^n\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}(1-x_i^{\alpha})^{\theta-1}\frac{1}{\Gamma(b_4)a_4^{\,b_4}}\theta^{b_4-1}e^{\frac{-\theta}{a_4}}$$

$$\Rightarrow\pi(\theta\,|\,\underline{x})=K\theta^{n+b_4-1}e^{-\theta\left(\frac{1}{a_4}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)}$$

$$\Rightarrow\pi(\theta\,|\,\underline{x})=K\theta^{n+b_4-1}e^{-\theta\beta_4} \qquad (10.19)$$

where K is independent of θ, and $\beta_4=\left(\frac{1}{a_4}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)$

And $$K^{-1}=\int_0^{\infty}\theta^{n+b_4-1}e^{-\theta\beta_4}\,d\theta=\frac{\Gamma(n+b_4)}{\beta_4^{\,n+a_4}}$$

Therefore from (10.19) we have

$$\pi_{ER}(\theta\,|\,\underline{x})=\frac{\beta_4^{\,n+b_4}}{\Gamma(n+b_4)}\ \theta^{n+b_4-1}e^{-\theta\beta_4}\qquad \theta>0 \qquad (10.20)$$

which is the pdf of gamma distribution with parameters $\gamma_4=(n+b_4)$ & $\beta_4=\left(\frac{1}{a_4}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)$

10.4.2. Posterior distribution under Exponential Prior

The pdf of Exponential distribution with hyper parameter c_5 is given as

$$g(\theta)=\frac{1}{c_5}e^{-\frac{\theta}{c_5}}\ , \qquad 0<\theta<\infty;\ c_5>0 \qquad (10.21)$$

Thus posterior distribution of θ is given as

$$\pi(\theta\,|\,\underline{x})=K\theta^{n+1-1}e^{-\theta\left(\frac{1}{c_5}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)}$$

$$\Rightarrow\pi(\theta\,|\,\underline{x})=K\theta^{n+1-1}e^{-\theta\beta_5} \qquad (10.22)$$

Where K is independent of θ, $\beta_5=\left(\frac{1}{c_5}-\sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)$

and $K^{-1} = \int_0^\infty \theta^{n+1-1} e^{-\theta\beta_5} \, d\theta$

$$\Rightarrow \qquad K^{-1} = \frac{\Gamma(n+1)}{\beta_5^{\,n+1}}$$

Therefore from (10.22) we have

$$\pi_E(\theta \mid \underline{x}) = \frac{\beta_5^{\,n+1}}{\Gamma(n+1)} \; \theta^{n+1-1} e^{-\theta\beta_5} \qquad \theta > 0 \tag{10.23}$$

which is the pdf of gamma distribution with parameters $\gamma_5 = (n+1)$ & $\beta_5 = \left(\frac{1}{c_5} - \sum_{i=1}^{n} \ln(1 - x_i^{\,\alpha})\right)$

10.4.3. Posterior distribution under Extension of Jeffrey's prior

Al-kutubi (2005) introduced new extension of Jeffrey's prior given as

$$g(\theta) = [I(\theta)]^m \quad ; m \in R^+ \tag{10.24}$$

$$[I(\theta)] = -E\left[\frac{\partial^2}{\partial\theta^2} \log L(\theta, x)\right]$$

Thus extension of Jeffrey's prior for Minimax distribution is $g(\theta) \propto \left(\frac{n}{\theta^2}\right)^m$ and the posterior distribution under extension of Jeffrey's prior is given by

$$\pi(\theta \mid \underline{x}) = K\theta^{n-2m+1-1} e^{-\theta \sum_{i=1}^{n} \ln(1 - x_i^{\,\alpha})^{-1}}$$

$$\Rightarrow \pi(\theta \mid \underline{x}) = K\theta^{n-2m+1-1} e^{-\theta\beta_6} \tag{10.25}$$

Where K is independent of θ, $\beta_6 = \sum_{i=1}^{n} \ln(1 - x_i^{\,\alpha})^{-1}$

and $K^{-1} = \int_0^\infty \theta^{n-2m+1-1} e^{-\theta\beta_6} \, d\theta$

$$\Rightarrow \qquad K^{-1} = \frac{\Gamma(n-2m+1)}{\beta_6^{\,n-2m+1}}$$

Therefore from (10.25) we have

$$\pi_{EJ}(\theta \mid \underline{x}) = \frac{\beta_6^{\,n-2m+1}}{\Gamma(n-2m+1)} \; \theta^{n-2m+1-1} e^{-\theta\beta_6} \qquad \theta > 0 \tag{10.26}$$

which is the pdf of gamma distribution with parameters $\gamma_6 = (n-2m+1)$ & $\beta_6 = \sum_{i=1}^{n} \ln(1 - x_i^{\,\alpha})^{-1}$

Thus Baye's estimators for θ under Erlang prior, Exponential Prior, and extension of Jeffery's prior are

$$\hat{\theta}_{E_RP} = \frac{n+b_4}{\left(\frac{1}{a_4} - \sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)} \; ; \hat{\theta}_{E_XP} = \frac{(n+1)}{\left(\frac{1}{c_5} - \sum_{i=1}^{n}\ln(1-x_i^{\alpha})\right)} \; ; \hat{\theta}_{J_XP} = \frac{(n-2m+1)}{\left(\sum_{i=1}^{n}\ln(1-x_i^{\alpha})^{-1}\right)}$$

10.5. Comparison of Single Priors with Respect to Posterior Variances

The variances of the posterior distribution under all assumed single priors is given by

$$V(\theta \mid X) = \frac{\gamma_i}{\beta_i^{2}}; i = 4,5,6. \tag{10.27}$$

10.6. The Prior Predictive Distribution

The prior predictive distribution of an unobserved data value is the product of the prior for θ and the single variable pdf, integrating out this parameter. This makes intuitive sense as uncertainty in θ is averaged out to reveal a distribution for the data point. It is defined as $\pi(y) = \int_0^{\infty} \pi(\theta) f(y,\theta) d\theta$ where Y is the random variable of the model with unknown parameter θ.

$$f(y;\theta,\alpha) = \alpha\theta y^{\alpha-1}(1-y^{\alpha})^{\theta-1} \quad ; 0<y<1 \;, \quad \theta,\alpha>0$$

Thus the Prior Predictive Distribution using Gamma-Exponential Prior is given as

$$\pi(y) = \int_0^{\infty} g_{11}(\theta) f(y;\theta,\alpha) d\theta$$

$$= \int_0^{\infty} \theta^{a_1-1} e^{-\theta(b_1+c_1)} \alpha\theta y^{\alpha-1}(1-y^{\alpha})^{\theta-1} d\theta \qquad = \frac{\alpha y^{\alpha-1}}{(1-y^{\alpha})} \int_0^{\infty} \theta^{a_1+1-1} e^{-\theta(b_1+c_1-\ln(1-y^{\alpha}))} d\theta$$

$$\Rightarrow \pi(y) = \frac{\alpha y^{\alpha-1}\Gamma(a_1+1)}{(1-y^{\alpha})\left[b_1+c_1-\ln(1-y^{\alpha})\right]^{a_1+1}} \tag{10.28}$$

Prior Predictive Distribution using Chi-Square-Exponential Prior is

$$\pi(y) = \int_0^{\infty} g_{22}(\theta) f(y;\theta,\alpha) d\theta$$

$$= \frac{\alpha y^{\alpha-1}}{(1-y^{\alpha})} \int_0^{\infty} \theta^{\left(\frac{a_2}{2}+1\right)-1} e^{-\theta\left(\frac{1}{c_1}+\frac{1}{2}-\ln(1-y^{\alpha})\right)} d\theta$$

$$\Rightarrow \pi(y) = \frac{\alpha y^{\alpha-1}\Gamma(a_2/2+1)}{(1-y^{\alpha})\left[\frac{1}{c_1}+\frac{1}{2}-\ln(1-y^{\alpha})\right]^{\frac{a_2}{2}+1}} \tag{10.29}$$

Prior Predictive Distribution using Gamma Chi-Square Prior is

$$\pi(y) = \int_0^{\infty} g_{33}(\theta) f(y;\theta,\alpha)\,d\theta$$

$$= \frac{\alpha y^{\alpha-1}}{(1-y^{\alpha})}\int_0^{\infty} \theta^{\left(a_3+\frac{a_4}{2}\right)-1} e^{-\theta\left(b_2+\frac{1}{2}-\ln(1-y^{\alpha})\right)} d\theta$$

$$\Rightarrow \pi(y) = \frac{\alpha y^{\alpha-1}\Gamma(a_3+a_4/2)}{(1-y^{\alpha})\left[b_2+\frac{1}{2}-\ln(1-y^{\alpha})\right]^{a_3+\frac{a_4}{2}}} \tag{10.30}$$

11. Simulation Study and Data Analysis

11.1. Simulation Study

In our simulation study, we choose a sample size of n=25, 50 and 100 to represent small, medium and large data set. The shape parameter θ is estimated for minimax distribution by using Bayesian method of estimation under various types of priors. For the shape parameter θ we have considered $\theta = 0.5, 1.5$ and 2.5. The shape parameter α has been fixed at 1.0. The value of Jeffrey's extension were m= 2 and the values of hyper parameters were ai, bi and ci =1.0, 1.5, 2.0 and 2.5. This was iterated 10000 times and the shape parameter θ for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes by using various types of priors. We used *minimax* package in R software for simulation study. The results are presented in tables given below:

Table 11.1: Posterior Mean and Posterior Variance of a Minimax Distribution using different priors with n=25.

α	θ	*Hyper Parameters ai=bi=ci*	*Mean/P.V*	*Gamma Exponential distribution*	*Chi-Square Exponential distribution*	*Gamma Chi-Square Distribution*	*Erlang distribution*	*Exponential distribution*	*Extension Jeffrey's prior m=2*
1.0	0.5	1.0	**Mean**	0.40113	0.39647	0.39647	0.40741	0.40741	0.35022
			post.var	0.01147	0.01149	0.01149	0.01197	0.01197	0.01058
		1.5	**Mean**	0.61789	0.62721	0.62667	0.65344	0.64111	0.55154
			post.var	0.00793	0.00822	0.00814	0.00862	0.00845	0.00732
		2.0	**Mean**	0.52015	0.53160	0.53562	0.55775	0.53709	0.45920
			post.var	0.00754	0.00805	0.00794	0.00851	0.00819	0.00706
		2.5	**Mean**	0.49435	0.50942	0.51745	0.53891	0.50952	0.43454
			post.var	0.01036	0.01168	0.01132	0.01249	0.01181	0.01017

1.0	1.5	1.0	**Mean**	1.05891	1.06014	1.06014	1.10387	1.10387	0.97546
			post.var	0.08806	0.09162	0.09162	0.099280	0.099280	0.09544
		1.5	**Mean**	0.98136	1.02304	1.00948	1.07417	1.05391	0.916537
			post.var	0.07103	0.08425	0.07825	0.09189	0.09015	0.08265
		2.0	**Mean**	1.51092	1.74849	1.64937	1.87893	1.80933	1.58617
			post.var	0.10792	0.15826	0.13171	0.17796	0.17137	0.15753
		2.5	**Mean**	1.54689	1.91919	1.75882	2.08687	1.97305	1.72176
			post.var	0.05262	0.07457	0.06376	0.08246	0.07796	0.06896
1.0	2.5	1.0	**Mean**	1.60174	1.62087	1.62087	1.70690	1.70690	1.54578
			post.var	0.18236	0.19482	0.19482	0.21722	0.21722	0.22264
		1.5	**Mean**	2.50602	2.94583	2.74165	3.21556	3.15489	2.90448
			post.var	0.23493	0.33347	0.28361	0.38585	0.37857	0.37882
		2.0	**Mean**	1.61116	1.88979	1.76955	2.03648	1.96106	1.72439
			post.var	0.20890	0.37124	0.27723	0.43607	0.41992	0.40517
		2.5	**Mean**	2.00819	2.73611	2.37303	3.02400	2.85906	2.53051
			post.var	0.13142	0.24431	0.17858	0.28255	0.26714	0.24555

Table 11.2: Posterior Mean and Posterior Variance of a Minimax Distribution using different priors with n=50.

α	θ	*Hyper Parameters ai=bi=ci*	*Mean/P.V*	*Gamma Exponential distribution*	*Chi-Square Exponential distribution*	*Gamma Chi-Square Distribution*	*Erlang distribution*	*Exponential distribution*	*Extension Jeffrey's prior m=2*
1.0	0.5	1.0	**Mean**	0.51978	0.51733	0.51733	0.52514	0.52514	0.48898
			post.var	0.00438	0.00437	0.00437	0.00446	0.00446	0.00419
		1.5	**Mean**	0.37933	0.37893	0.38029	0.38597	0.38222	0.35401
			post.var	0.00791	0.00816	0.00807	0.00839	0.00831	0.00779
		2.0	**Mean**	0.42919	0.43162	0.43456	0.44195	0.43345	0.40116
			post.var	0.00319	0.00329	0.00327	0.00338	0.00331	0.00308
		2.5	**Mean**	0.42358	0.42764	0.43258	0.43991	0.42734	0.39515
			post.var	0.00393	0.00412	0.00409	0.00426	0.00414	0.00384
1.0	1.5	1.0	**Mean**	1.35454	1.35931	1.35931	1.39149	1.39149	1.31833
			post.var	0.06416	0.06585	0.06585	0.06897	0.06897	0.06851
		1.5	**Mean**	1.29824	1.34132	1.32535	1.37937	1.36598	1.28173
			post.var	0.05916	0.06627	0.06307	0.06975	0.06907	0.06690
		2.0	**Mean**	1.57010	1.69329	1.64458	1.75563	1.72187	1.61407
			post.var	0.06242	0.07625	0.06945	0.08084	0.07929	0.07604
		2.5	**Mean**	1.59508	1.77867	1.70637	1.85423	1.80125	1.68377
			post.var	0.03542	0.04330	0.03959	0.04568	0.04437	0.04188
1.0	2.5	1.0	**Mean**	2.53915	2.57845	2.57845	2.67219	2.67219	2.59878
			post.var	0.11671	0.12129	0.12129	0.12873	0.12873	0.13152
		1.5	**Mean**	2.16447	2.31102	2.24847	2.39981	2.37652	2.26034
			post.var	0.09920	0.11562	0.10799	0.12313	0.12193	0.12007
		2.0	**Mean**	2.10319	2.34760	2.23903	2.45002	2.40290	2.26787
			post.var	0.10473	0.13717	0.12040	0.14741	0.14457	0.14062
		2.5	**Mean**	1.84159	2.09973	1.98997	2.19592	2.13318	1.99932
			post.var	0.07031	0.09501	0.08224	0.10165	0.09875	0.09430

Table 11.3: Posterior Mean and Posterior Variance of a Minimax Distribution using different priors with n=100

α	θ	*Hyper Parameters ai=bi=ci*	*Mean/P.V*	*Gamma Exponential distribution*	*Chi-Square Exponential distribution*	*Gamma Chi-Square distribution*	*Erlang distribution*	*Exponential distribution*	*Extension Jeffrey's prior m=2*
1.0	0.5	1.0	**Mean**	0.48676	0.48551	0.48551	0.48911	0.48911	0.47202
			post.var	0.00222	0.00221	0.00221	0.00224	0.00224	0.00217
		1.5	**Mean**	0.59520	0.59723	0.59724	0.60346	0.60049	0.57900
			post.var	0.00225	0.00228	0.00227	0.00230	0.00229	0.00221

		2.0	**Mean**	0.44911	0.45066	0.45209	0.45614	0.45167	0.43475
			post.var	0.00196	0.00199	0.00198	0.00202	0.00200	0.00193
		2.5	**Mean**	0.50824	0.51246	0.51458	0.52010	0.51249	0.49319
			post.var	0.00312	0.00323	0.00320	0.00329	0.00324	0.00313
1.0	1.5	1.0	**Mean**	1.27917	1.28095	1.28095	1.29558	1.29558	1.26044
			post.var	0.01809	0.01824	0.01824	0.01858	0.01858	0.01834
		1.5	**Mean**	1.44615	1.47397	1.46344	1.49588	1.48851	1.44375
			post.var	0.02147	0.02249	0.02205	0.02300	0.02288	0.02243
		2.0	**Mean**	1.43802	1.48681	1.46909	1.51266	1.49783	1.44926
			post.var	0.02383	0.02592	0.02497	0.02661	0.02635	0.02572
		2.5	**Mean**	1.38765	1.45128	1.42975	1.47981	1.45815	1.40854
			post.var	0.02129	0.02376	0.02264	0.02443	0.02407	0.02341
1.0	2.5	1.0	**Mean**	2.46465	2.48274	2.48274	2.52629	2.52629	2.48849
			post.var	0.06065	0.06186	0.06186	0.06374	0.06374	0.06441
		1.5	**Mean**	2.37087	2.45864	2.4216	2.50754	2.49519	2.4365
			post.var	0.04902	0.05283	0.05112	0.05446	0.05419	0.05369
		2.0	**Mean**	2.30885	2.45277	2.38999	2.50751	2.48292	2.41426
			post.var	0.03674	0.04091	0.03893	0.04215	0.04174	0.04091
		2.5	**Mean**	2.18802	2.36864	2.29118	2.42627	2.39076	2.31803
			post.var	0.06082	0.07416	0.06738	0.07714	0.07602	0.07463

11.2. A Real Data Example

In this section, we analyze a data set from Abuammoh et al. (1994), which represent the lifetime in days of 40 patients suffering from leukemia from one of the Ministry of Health Hospitals in Saudi Arabia and the ordered life time (in days) are:

115 ,181 ,255, 418 ,441 ,461 ,516 ,739 ,743 ,789, 807, 865 ,924, 983 ,1024 ,1062, 1063, 1165, 1191, 1222 ,1222 ,1251, 1277 ,1290 ,1357, 1369, 1408, 1455, 1478 ,1549 ,1578 ,1578 ,1599 ,1603 ,1605, 1696 ,1735 ,1799, 1815 ,1852

Table 11.4: Posterior Mean and Posterior Variance of a Minimax Distribution using different priors with Real Data

α	θ	*Hyper Parameters ai=bi=ci*	*Mean/P.V*	*Gamma Exponential distribution*	*Chi-Square Exponential distribution*	*Gamma Chi-Square Distribution*	*Erlang distribution*	*Exponential distribution*	*Extension Jeffrey's prior m=2*
1.0	0.5	1.0	**Mean**	0.54317	0.54012	0.54012	0.55046	0.55046	0.50351
			post.var	0.00493	0.00492	0.00492	0.00504	0.00504	0.00465
		1.5	**Mean**	0.40864	0.40863	0.41022	0.41825	0.41321	0.37542
			post.var	0.00937	0.00973	0.00960	0.01007	0.00994	0.00916
		2.0	**Mean**	0.41205	0.41443	0.41820	0.42669	0.41654	0.37782
			post.var	0.00764	0.00810	0.00796	0.00842	0.00822	0.00752
		2.5	**Mean**	0.46394	0.47139	0.47708	0.48847	0.47123	0.42722
			post.var	0.00537	0.00573	0.00565	0.00597	0.00576	0.00525
1.0	1.5	1.0	**Mean**	1.74973	1.76608	1.76608	1.82773	1.82773	1.72637
			post.var	0.04535	0.04633	0.04633	0.04852	0.04852	0.04697
		1.5	**Mean**	1.27494	1.32662	1.30743	1.37339	1.35684	1.25209
			post.var	0.04131	0.04569	0.04378	0.04814	0.04756	0.04493
		2.0	**Mean**	1.39842	1.51662	1.47194	1.58289	1.54520	1.42123
			post.var	0.02995	0.03455	0.03250	0.03645	0.03558	0.03307
		2.5	**Mean**	1.25622	1.38741	1.34299	1.45389	1.40258	1.28331
			post.var	0.05669	0.07611	0.06637	0.08190	0.07901	0.07387

1.0	2.5	1.0	**Mean**	1.81233	1.83069	1.83069	1.89615	1.89615	1.79413
			post.var	0.09641	0.100031	0.100031	0.10649	0.10649	0.10670
		1.5	**Mean**	2.541085	2.81068	2.69052	2.96465	2.92893	2.77536
			post.var	0.09089	0.10679	0.09943	0.11454	0.11316	0.10967
		2.0	**Mean**	1.84872	2.07928	1.97942	2.18541	2.13338	1.97667
			post.var	0.11092	0.15134	0.13022	0.16489	0.16097	0.15482
		2.5	**Mean**	1.61261	1.85354	1.75530	1.95359	1.88465	1.73264
			post.var	0.06620	0.09145	0.07849	0.09882	0.09533	0.08945

The posterior mean and posterior variance under all the assumed priors is calculated by assuming the different values of hyper parameters. From table 11.1 to 11.4, it is clear that the posterior variance under the double prior Gamma -Exponential distribution are less as compared to other assumed priors, which shows that this prior is efficient as compared to other priors and this less variation in posterior distribution helps in making more precise Bayesian estimation of the true unknown parameter θ of Minimax distribution.

12. Conclusion

In this paper, we addressed the problem of Bayesian estimation for the Minimax distribution under different priors each in the worked example as well as in the simulation study. From the results, we observe that in most cases, Bayesian Estimator under the double prior Gamma-Exponential distribution has the less posterior variance.

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