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Article

On Relating Vertex Covers and Dominating Sets in Simple Connected Graphs

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Abstract: In any graph, each vertex cover is a dominating set, but the converse is not true. This article characterizes the simple loop-free connected graphs for which each dominating set is a vertex cover. This characterization holds for all simple graphs since a disconnected graph is a union of its components.

Keywords: Simple graph; Vertex cover; Dominating set; Bipartite graph; Pendant vertex.

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1. Introduction

In Graph Theory, vertex covering and domination are two topics on which hundreds of publications have been brought out. Some seminal works on the vertex cover problem are in [4, 5, 7, 8, 9, 16, 17, 18, 21]. [1, 2, 10, 13, 14, 15, 20] are excellent research reports on domination, the study of which seems to have begun in 1960 [3]. But almost all the research seems to be for minimum vertex covers or minimum dominating sets, with algorithms being accorded prime status. To the best of our knowledge, there has been no attempt at characterization-oriented study of interplay between vertex covers and dominating sets, and this is a motivation for this article.

First, some preliminary theory. A substantial portion of this can be found in [4, 6, 11]. The presentation here is purely for ready reference. If V is a finite nonempty set, then 2^V denotes its *power set* [19] – that is, the set of all the subsets of V (including the empty set ϕ); and 2^{V*} denotes the set of all nonempty subsets of V – that is, $2^{V*} = 2^V - \{\phi\}$. And $2^{V*} - \{V\}$ is the set of all nonempty proper subsets of V. The *cardinality* (or, *size*) of a finite set V is denoted by $|V|$, and is the number of elements in V.

A *simple graph* is an ordered pair G = (V, E) where V is a nonempty finite set and $E \subset 2^{V*}$ such that (i) if $y \in V$ then $y \in X$ for some $X \in E$, and (ii) $|X| \leq 2$ for each $X \in E$. This definition does not permit any isolated vertices [11]. The sets V and E are, respectively, the *vertex set* and the *edge set* of the graph G. Each element of V is a *vertex* in G and each member of E is an *edge* in G. The integers $|V|$ and $|E|$ are, respectively, the *order* (= the number of vertices) and the *rank* (= the number of edges) of G. A *loop* is an edge X with $|X| = 1$; more precisely, a loop at a vertex y is just the set {y} in E. G is *loop-free* if $|X| = 2$ for each $X \in E$.

If x, $y \in V$ are distinct, then x and y are *adjacent* if $\{x, y\} \in E$; in this case x and y are *neighbors* in G. If $\{x, y\} \in E$ then x and y are the *end points* of the edge {x, y}. Let $S \in 2^{V*} - \{V\}$. The set N(S) = $\{y \in V \mid y$ is adjacent to some $x \in S\}$ is the *open neighborhood* of S; the set $C(S) = N(S) \cup S$ is the *closed neighborhood* of S. Open neighborhoods will simply be referred to as neighborhoods. In particular, when S is a singleton, say S = {x}, then N(S) is denoted by N(x), the neighborhood of x, and C(S) by C(x), the closed neighborhood of x.

The *degree* of a vertex x in G is denoted by dx (or, dx(G), if G needs mention) and is defined as $dx = |N(x)|$. A vertex y with dy = 1 is a *pendant vertex* (or, a *leaf*). The *largest degree* of G is denoted by Δ(G), and is defined as $\Delta(G) = \max\{dx \mid x \in V\}$.

A *path* in G between two distinct vertices x and y is a sequence x, $z_1, \ldots . z_k$, y of distinct vertices in G such that (i) x is adjacent to z_1; (ii) y is adjacent to z_k; and (iii) z_j is adjacent to z_{j+1} for j = 1 through $k - 1$. Obviously, if x and y are adjacent, then the edge joining x and y is a path between them. A vertex x is *connected to* a vertex y if there is a path between x and y. G is a *connected graph* if x is connected to y whenever x and y are distinct vertices in G.

G = (V, E) is *complete* if each vertex in G is adjacent to every other vertex in G. If $W \in 2^{V*}$ then W is *independent* if no two vertices in W are adjacent.

G is *bipartite* if there is a partitioning $V = A \cup B$ (that is, $A \cap B = \phi$) such that (i) A and B are independent; and (ii) each edge in G has one end point in A and the other end point in B. In this case, G is also written G = [A, B]. A bipartite graph G = [A, B] is *complete bipartite* if every vertex of A is adjacent to every vertex of B. If G = [A, B] is complete bipartite then G is also written $G = K_{p,q}$ where $p = |A|$ and $q = |B|$.

S *covers* an edge $\{x, y\}$ if $S \cap \{x, y\} \neq \phi$. S is a *vertex cover* (for G) if $X \in E \Longrightarrow S \cap X \neq \phi$ (that is, S covers every edge in G). Additionally, if no proper subset of S is a vertex cover, then S is a *minimal vertex cover*. S is a *minimum vertex cover* if (i) S is a vertex cover and (ii) $|T| \geq |S|$ for every vertex cover T. If S is a minimum vertex cover, then the positive integer $|S|$ is the *vertex cover number* of G.

Let $D \subset V$. Then D is a *dominating set* if for each $x \in V$, either $x \in D$ or x is adjacent to some y $\in D$. If D is a dominating set and no proper subset of D is a dominating set, then D is a *minimal dominating set*. Additionally, if $|M| \geq |D|$ for every dominating set M in G, then D is a *minimum dominating set*. If D is a minimum dominating set then $|D|$ is the *dominating number* of G.

If $\beta(G)$ and $\gamma(G)$ denote, respectively, the vertex cover number and the dominating number of G, then $\beta(G) \geq \gamma(G)$.

1.1 is a known result; its proof is straightforward [12]. 1.2 is a counterexample to the converse of 1.1. The central problem of this article stems from 1.2.

1.1: Proposition. Every vertex cover is a dominating set.

1.2: Example. The converse of 2.1 is not true. If G is a complete graph on n (≥ 3) vertices, then for each $x \in V$, $\{x\}$ is a dominating set but not a vertex cover.

Central problem of this article: Characterize the simple loop-free connected graphs for which every dominating set is a vertex cover.

This is answered in 2.3 (section 2). All the graphs in the coming discussions are assumed simple, loop-free, connected, with at least two vertices and one edge.

2. Results and Discussion

2.1: Proposition. Let $G = K_{1, m}$. Then every dominating set of G is a vertex cover.

Proof. Let V be the vertex set of G. Note that $|V| = m + 1$.The conclusion is obvious when m = 1, and so let $m \geq 2$. There is a unique $x \in V$ such that $dx = m$, and. Let D be a dominating set. Suppose $x \in D$. Then clearly every edge is covered by D.

Suppose $x \notin D$. Then $D \subset V - \{x\}$. Write $V - \{x\} = \{y_1, . . ., y_m\}$. Clearly $V - \{x\}$ is independent. If some $y_j \notin D$ then y_j has no neighbor in D. But this would contradict the dominating nature of D. So $V - \{x\} \subset D$. Then D ($= V - \{x\}$) is a vertex cover.

2.2: Proposition. Let G = (V, E) have the property that each dominating set is a vertex cover. Then:

(a) if $\{x, y\} \in E$ then either x or y is a pendant vertex;

(b) there is a $z \in V$ such that $dz = |V| - 1$; and z is unique if $|V| \geq 3$;

(c) the set $V - \{z\}$ is independent (where z is as in (b)); and

(d) $G = K_{1,m}$ where $m = |V - \{z\}|$.

Proof. (a) Suppose there is $\{x, y\} \in E$ such that neither x nor y is a pendant vertex. Let $V - \{x, y\} = D$. Then D is a dominating set because each of x and y has a neighbor in D. Also, D is not a vertex cover because D does not cover the edge $\{x, y\}$. But this contradicts the hypothesis. Thus (a) follows.

(b) Let $z \in V$ be such that $dz = \Delta(G)$. If $dz = 1$, then $|V| = 2$ (owing to the connectedness of G), and (b) is immediate. So assume $dz > 1$. Then $|V| \geq 3$. Suppose some y ($\in V$) is not a neighbor of z. Since G is connected, there is a path – call it P – between z and y. Since z and y are not adjacent, there is an intermediate vertex, say b, in P such that b and z are adjacent. Since b is an intermediate vertex in P, there is a vertex c in P such that (i) c is distinct form b and x, and (ii) c is adjacent to b. Then $db > 1$, and so the edge $\{b, z\}$ has no pendant vertices. But this cannot happen in G because of (a). Consequently, every $y \in V - \{z\}$ is a neighbor of z, whence $dz = n - 1$, where $n = |V|$.

Next, with $|V| \geq 3$, suppose there are two distinct vertices z_1 and z_2 in V with $dz_1 = dz_2 = n - 1$. Note that $n - 1 \geq 2$. Then $\{z_1, z_2\} \in E$. But then the edge $\{z_1, z_2\}$ is without a pendant vertex, contradicting (a). This completes (b).

(c) From (b), it is clear that $\{z\}$ is a dominating set – and so by hypothesis, $\{z\}$ is a vertex cover. Then $V - \{z\}$ is independent.

(d) Let z be as in (b). Setting $A = \{z\}$ and $B = V - \{z\}$, (d) follows at once.

2.3: Proposition. Let G be a simple loop-free connected graph. Then each dominating set in G is a vertex cover if and only if $G = K_{1,m}$, where $m = |V| - 1$.

Proof. Consequence of 2.1 and 2.2.

2.4: Proposition. Let G be a simple loop-free connected graph. Suppose each dominating set in G is a vertex cover. Then:

(a) each minimal dominating set in G is a minimal vertex cover; and

(b) each minimal vertex cover in G is a minimal dominating set.

Proof. (a) Let D be a minimal dominating set. Then by hypothesis, D is a vertex cover. Let M be a minimal vertex cover such that $M \subset D$. Then M is a dominating set (by 1.1). Since D is a minimal dominating set, it follows that $M = D$, whence D is a minimal vertex cover.

(b) Let D be a minimal vertex cover. Then by 1.1, D is a dominating set. Let M be a minimal dominating set such that $M \subset D$. Then M is a vertex cover (by hypothesis). Since D is a minimal vertex cover, it follows that $M = D$, whence D is a minimal dominating set.

2.5: Proposition. Let G be a simple loop-free connected graph. Suppose each minimal dominating set in G is a minimal vertex cover. Then each dominating set in G is a vertex cover.

Proof. Let D be a given dominating set. Then there is a minimal dominating set $D_1 \subset D$. By hypothesis, D_1 is a minimal vertex cover, whence D is a vertex cover.

2.6: Proposition. Let $\boldsymbol{\beta}$ (G) denote the vertex cover number of the graph G. Then $\boldsymbol{\beta}(G) = 1$ if and only if $G = K_{1,\, m}$.

Proof. Assume $\boldsymbol{\beta}$ (G) = 1. Let S = {x} be a vertex cover. Then $V - \{x\}$ is independent and every vertex in $V - \{x\}$ is adjacent to x, whence $G = K_{1,\, m}$ (where $m = |V - \{x\}|$).

Conversely, suppose $G = K_{1,\, m}$. For m = 1 the conclusion is immediate. So let $m \geq 2$. Then dx = m for a unique $x \in V$, by 2.2(b). So{x} is a vertex cover, whence $\boldsymbol{\beta}(G) = 1$.

2.7: Proposition. For a simple loop-free connected graph G, the following are equivalent:

(a) $G = K_{1,\, m}$;

(b) each dominating set in G is a vertex cover; and

(c) $\boldsymbol{\beta}(G) = 1$.

Proof. Consequence of 2.3 and 2.6.

3. Conclusions

1) The only class of simple loop-free connected graphs for which each dominating set is a vertex cover is the class $\boldsymbol{\mathcal{K}_1}$ of the graphs $K_{1,\, m}$ where m is a positive integer (see 2.3).

2) Hence, if a graph G is in the class $\boldsymbol{\mathcal{K}_1}$,every algorithm that enumerates the minimal vertex covers of G will also enumerate the minimal dominating sets of G, and vice-versa.

3) If a graph G is not in the class $\boldsymbol{\mathcal{K}_1}$, then G has a dominating set that is not a vertex cover.

3)$\boldsymbol{\beta}(G) = \boldsymbol{\gamma}(G) = 1$ if and only if G is in the class $\boldsymbol{\mathcal{K}_1}$ (see 2.7).

Acknowledgments

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