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Article

Analytical Expression to the Steady Viscous Flow of a Micropolar Fluid Driven by Injection between Two Porous Disks

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Abstract: In this paper a steady, laminar, incompressible and two-dimensional flow of a micropolar fluid between two porous coaxial disks is being considered. Using the micropolar model we have described the working fluid. The governing equations of motion are reduced to a set of non-linear coupled ordinary differential equations using Berman's similarity transformation. Homotopy Analysis Method (HAM) is employed to compute an approximation to the solution of the system of nonlinear differential equations governing the problem. It has been attempted to show the capabilities and wide-range applications of the Homotopy analysis method. The obtained solutions admit a remarkable accuracy.

Keywords: Micropolar; Porous Disks; Microrotation; Skin friction; Homotopy analysis method

Mathematics Subject Classification (2010): 34E, 35k20 and 68U20

1. Introduction

The micropolar fluids are those which consist of bar-like elements. For example, the micropolar model has been used to describe the flow of liquid crystals which are made up of dumbbell molecules. Other examples of such fluids are polymeric fluids and real fluids with suspensions, fluids with some polymeric additives, suspension solutions, and nemotogenic and smectogenic liquid crystals. The need to model the non-Newtonian flow of fluids containing rotating micro-constituents was the basic idea for the origin of micropolar fluids. This theory of micropolar fluids was originally developed by Eringen [1]. This study introduces some new material parameters, an additional independent vector field - the microrotation - and new constitutive equations that must be solved simultaneously with the usual Newtonian flow equations. Subsequent studies showed that the model can be successfully applied to a wide range of applications including blood flow, lubricants, porous media, turbulent shear flows, and flow in capillaries and micro channels. More recent studies can be found on [2].

Different authors have considered the effects of porous boundaries on steady, laminar and incompressible flow for different geometrical shapes. Two-dimensional flows in a porous channel has been studied by several authors, e.g. Berman [3], Cox [4], Brady [5], Terrill and Shrestha [6,7], Terrill [8], and Robinson [9]. Berman [3] considered flow which was driven by suction or injection, while Cox [4] examined by considering channel with one porous wall and other non-porous but was accelerating. Terrill and Shrestha [7] considered the laminar flow for small Reynolds numbers through parallel and uniformly porous walls of different permeability with different suction and injection normal velocities at the walls. They obtained the series solution using perturbation method and compared it with the numerical solution by calculating the skin friction at the lower and the upper wall. Terrill [8] obtained the complete solution of the laminar flow of a fluid in a uniformly porous channel with large injection by the method of inner and outer expansions and included the viscous layer. The resulting series solutions were confirmed by numerical results. In all these cases the Navier–Stokes equations are reduced to ordinary non-linear differential equations of third order for which approximate solutions are obtained by a mixture of analytical and numerical methods. Rasmussen [10] examined the steady viscous flow between two porous disks. The flow was driven by injection or suction at both walls. He used an extension of Berman's [3] similarity transformation to reduce governing equations to a set of non-linear coupled ordinary differential equations in dimensionless form. The boundary value problem was converted into initial value problem and shooting method was used to solve it numerically. He only considered the symmetric case in which there was equal blowing or suction at both the disks. All the above researchers have done their work for Newtonian fluids.

Guram and Anwar [11] considered the problem of steady, laminar and incompressible flow of a micropolar fluid due to a rotating disc with uniform suction and injection. The equations of motion were reduced to dimensionless form and then solved by Gauss–Siedel iterative procedure with Simpson's rule. Kelson et al. [12] analyzed self-similar boundary layer flow of a micropolar fluid in a porous channel. The flow was driven by uniform mass transfer through the channel walls. They used Berman [3] type similarity transformation to reduce governing equations of motion to a set of non-linear coupled ordinary differential equations which were then solved for large mass transfer via a perturbation analysis. Anwar Kamal and Siffat Hussain [13] examined the steady, incompressible and laminar flow of micropolar fluids inside an infinite channel. The flow was driven due to a surface velocity proportional to the stream wise coordinates. The governing equations were reduced to non-linear ordinary differential equations by using similarity transformation. These equations were then solved using numerical procedures which include SOR method. Anwar Kamal and Siffat Hussain [14] predicted the three-dimensional micropolar fluid motion caused by the stretching ofa surface. The resulting ordinary differential equations of motion were solved numerically using SOR method.

In this paper, the basic idea of the HAM is introduced and then, the nonlinear equation of the micropolar flow driven by injection between two porous disks is solved through the Homotopy analysis method. Because of the lack of extensive experimental data for micropolar flows, the primary goal of the present paper is to introduce new analytic solution for the theory.

2. Mathematical Formulation of the Problem

Consider two stationary porous disks of radius R_O located in the $z = -L$ and $z = L$ planes, respectively, and let the centers of the disks coincide with the axis r = 0 as shown in Fig. 1. The velocity components u_r and u_z are taken to be in the direction of $r-$ and $z-$axes, respectively. Fluid is injected at both the disks with constant velocities V_1 and V_2 at the lower and upper disk, respectively.

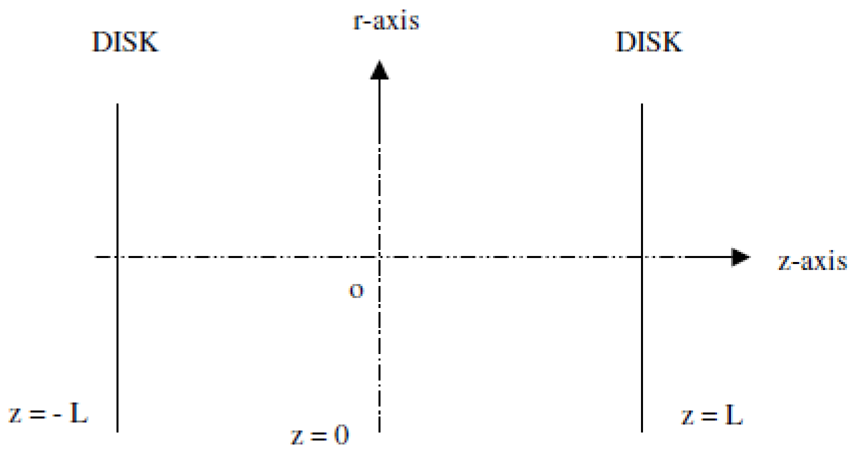

Fig. 1: Sketch showing location of disks

The governing equations of motion for the micropolar fluid given by Eringen [1] are as follows:

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{V}) = 0 \,, \tag{1}$$

$$(\lambda+2\mu+k)\nabla(\nabla.\underline{V})-(\mu+k)\nabla\times\nabla\times\underline{V}+k\nabla\times\underline{v}-\nabla\pi+\rho\underline{f}=\rho\underline{V} \quad , \tag{2}$$

$$(\alpha+\beta+\gamma)\nabla(\nabla.\underline{v})-\gamma(\nabla\times\nabla\times\underline{v})+k\nabla\times\underline{V}-2k\underline{v}+\rho\underline{l}=\rho j\dot{\underline{v}} \quad , \tag{3}$$

Where $\underline{V}$ is the fluid velocity vector , $\underline{v}$ the microrotation, ρ the density, π the pressure, $\underline{f}$ and $\underline{l}$ are body force and body-couple per unit mass , respectively, j the micro inertia, $\lambda,\mu,\alpha,\beta,\gamma,k$, the material constants (or viscosity coefficients), where dot signifies material derivative. Here the velocity vector $\underline{V}$ and $\underline{v}$ the microrotation vector are unknown.

The velocity and the spin rotation components are

$$\underline{V}=(u(r,z),0,w(r,z)) \ ,$$

$$\underline{v}=(0,\phi(r,z),0)\ , \tag{4}$$

Using the eqn.(2) in the governing equations of motion (1), we obtain

$$(\mu+k)\left(\frac{\partial^2 u}{\partial r^2}+\left(\frac{1}{r}\right)\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)-k\frac{\partial\phi}{\partial z}-\frac{\partial\pi}{\partial r}=\rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right) \tag{5}$$

$$(\mu+k)\left(\frac{\partial^2 w}{\partial r^2}+\left(\frac{1}{r}\right)\frac{\partial w}{\partial r}-+\frac{\partial^2 w}{\partial z^2}\right)-k\left(\frac{\partial\phi}{\partial r}+\frac{\phi}{r}\right)-\frac{\partial\pi}{\partial z}=\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right) \tag{6}$$

$$\begin{aligned}\gamma\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{\phi}{r^2}+\frac{\partial^2\phi}{\partial z^2}\right)+k\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right)-2k\phi \\ =\rho j\left(u\frac{\partial\phi}{\partial r}+w\frac{\partial\phi}{\partial z}\right) \quad ,\end{aligned} \tag{7}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{8}$$

On the disks the axial velocity u_z is prescribed and the radial velocity u_r is zero. Thus we have the following boundary conditions:

$$u_z(r,-L)=2V,u_z(r,L)=2V_2\,,u_r(R,-L)=0\ ,u_r(R,L)=0 \tag{9}$$

Where V_1 *and* V_2 are constants independent of r *and* θ.

We have to solve the eqns. (5)-(8) subject to boundary conditions eqns. (9). We let

$$u_r=-rF'(z),u_z=2F(z)\ ,\phi=-rG \tag{10}$$

If we use the eqn.(10) in eqn. (8), then we can find that equation of continuity is identically satisfied. Using the eqn.(10) in the eqns.(5) and (6) we get, the following equation after some simplifications and eliminating pressure term as:

$$(u+k)F^{iv}-kG''-2\rho FF''=0 \tag{11}$$

Where $\frac{\partial^2\pi}{\partial r\partial z}=0$ (12)

Now using the eqn. (10) in and eqn.(7) we found that

$$\gamma G'' + kF'' - 2kG - \rho j\left(F'G - 2FG'\right) = 0 \tag{13}$$

The dimensionless variables may be defined as

$$f(\lambda) = \frac{F(z)}{V},\ g(\lambda) = L^2 \frac{G(z)}{V} \tag{14}$$

Where $\lambda = \dfrac{Z}{L}$ and V is the larger of V_1 and V_2.

Using the eqn.(14) into the eqns. (11) and (13) we obtain that

$$f^{iv} - c_1 g'' - 2Rff''' = 0 \tag{15}$$

and

$$g'' + c_2\left(f'' - 2g\right) - c_3\left(f'g - 2fg'\right) = 0 \tag{16}$$

Where $R = \dfrac{\rho VL}{(\mu + k)}$ is the Reynolds number and

$$c_1 = \frac{k}{\mu + k},\ c_2 = \frac{kL^2}{\gamma},\ c_3 = \frac{\rho jLV}{\gamma}$$

are the dimensionless constants.

Integrating the eqn.(15) with respect to λ we get

$$f''' - c_1 g' - 2Rff'' + Rf'^2 = \beta \tag{17}$$

Where β is called the constant of integration and is known as pressure constant.

Using (14) in (10) the boundary condition reduces to

$$\begin{aligned} &f(1) = 1, && f(-1) = -1, \\ &f'(1) = 0, && f'(-1) = 0, \\ &g(1) = 0, && g(-1) = 0. \end{aligned} \tag{18}$$

In this paper we need to solve the eqns.(15) and (16) subject to the boundary conditions eqns. (18).

3. Analytical Expression for the Non-linear Differential Equations Using the Homotopy Analysis Method

This paper deals with a basic strong analytic tool for nonlinear problems, namely the Homotopy analysis method (HAM) which was generated by Liao [15], is employed to solve the nonlinear differential eqns. (11) - (14). The Homotopy analysis method is based on a basic concept in topology, i.e. Homotopy by Hilton [16] which is widely applied in numerical techniques as in [27-20]. Unlike perturbation techniques like [21], the Homotopy analysis method is independent of the small/large parameters. Unlike all other reported perturbation and non-perturbation techniques such as the artificial small parameter method [22], the δ-expansion method [23] and Adomian's decomposition method [24], the Homotopy analysis method provides us a simple way to adjust and control the convergence region and rate of approximation series. The Homotopy analysis method has been successfully applied to many nonlinear problems such as heat transfer [25], viscous flows [26], nonlinear oscillations [27], Thomas-Fermi's atom model [28], nonlinear water waves [29], etc.

Such varied successful applications of the Homotopy analysis method confirm its validity for nonlinear problems in science and engineering. The Homotopy analysis method is a good technique when compared to other perturbation methods. The existence of the auxiliary parameter h in the Homotopy analysis method provides us with a simple way to adjust and control the convergence region of the solution series.

In this paper we have used the Homotopy analysis method for the non-linear boundary value problem which is expressed in the eqns. (15) and (16) with the boundary conditions (18). And we have obtained the approximate analytical expression for the dimensionless stream function $f(\lambda)$ and dimensionless microrotation $g(\lambda)$ (see Appendix B) as follows:

$$f(\lambda)=-\frac{1}{2}\lambda^3+\frac{3}{2}\lambda-h\left(R\left(\frac{1}{280}\lambda^7-\frac{3}{40}\lambda^5\right)+\frac{39}{280}R\lambda^3-\frac{19}{280}R\lambda\right) \tag{19}$$

$$f'(\lambda)=-\frac{3}{2}\lambda^2+\frac{3}{2}-h\left(R\left(\frac{1}{40}\lambda^6-\frac{3}{8}\lambda^4\right)+\frac{117}{280}R\lambda^2-\frac{19}{280}R\right) \tag{20}$$

And

$$g(\lambda)=0-h\left(\frac{1}{2}\lambda^3-\frac{1}{2}\lambda\right)+h^2\left(\begin{array}{l} c_3\left(\frac{1}{28}\lambda^7-\frac{3}{40}\lambda^6-\frac{1}{4}\lambda^5+\frac{1}{4}\lambda^4+\frac{1}{4}\lambda^3-\frac{3}{8}\lambda^2\right) \\ -c_2\left(R\left(\frac{1}{280}\lambda^7-\frac{3}{40}\lambda^5\right)+\frac{39}{280}R\lambda^3-\frac{1}{20}\lambda^5+\frac{1}{6}\lambda^3\right) \\ \quad +A\lambda+B \end{array}\right) \tag{21}$$

Where

$$A=c_2\left(\frac{19}{280}R+\frac{7}{60}\right)-\frac{1}{28}c_3 \tag{22}$$

$$B=\frac{1}{5}c_3 \tag{23}$$

4. Results and Discussion

The analytical expressions for the steady viscous flow of a micropolar fluid driven by injection between two porous disks are obtained using the Homotopy Analysis method (HAM). In Fig.1 we can see the sketch showing the location of disks. The detailed computations regarding the axial and radial velocity distributions for various values of the Reynolds number R is shown in the following tables. Also we have calculated the skin friction at the lower and upper disks for various values of R which is given in Table: 4. The physics of the problem under consideration is explained through the detailed interpretation of the axial and radial velocity distributions for various Reynolds number R. In the discussion of skin friction we can see that there is no boundary layer on the disk in case of injection at the disk. The dimensional constants are fixed to take the values $c_1 = 1,\ c_2 = 1,\ c_3 = 0.1$.

Table: 1 Table values of dimensionless stream function $f(\lambda)$ for various values of λ at $R = -300$

λ	$f(\lambda)$
-1	-1
-0.8	-0.959908
-0.6	-0.839511
-0.4	-0.636316
-0.2	-0.358867
0	0
0.2	0.358867
0.4	0.636316
0.6	0.839511
0.8	0.959908
1	1

Table: 2 Table values of radial velocity $f'(\lambda)$ for various values of λ at $R = -300$

λ	$f'(\lambda)$
-1	0
-0.8	0.8011807
-0.6	1.602495
-0.4	2.403188
-0.2	3.204896
0	3.875678
0.2	3.20489
0.4	2.403188
0.6	1.602495
0.8	0.801180
1	0

Table: 3 Table values of dimensionless microrotation $g(\lambda)$ for various values of λ at $R=-300$

λ	$g(\lambda)$
-1	0
-0.8	1.13494
-0.6	1.63734
-0.4	1.63171
-0.2	1.11405
0	0.00003
0.2	-1.10351
0.4	-1.62376
0.6	-1.63145
0.8	-1.13056
1	0

Table: 4 Skin frictions at the lower and upper disks for various values of R

R	$f''(-1)$	$f''(1)$
-25	8.16728	-8.16728
-50	8.07857	-8.07857
-100	8.02971	-8.02971
s-200	8.01737	-8.01737
-300	8.01104	-8.01104
-400	8.00811	-8.00811
-500	8.00657	-8.00657

The axial velocity distributions are obtained in the Fig: 2, from which we can notice that the viscous layer coincides with the centre between the two disks in all cases of the Reynolds number R. The Radial velocity profiles are plotted and discussed for various values of R are shown in Fig.3. From Fig.3, we can infer that the radial velocity increases with the increase in the blowing at the disks. Also we notice that the radial velocity profiles are parabolic in nature. The microrotation of the micropolar fluids has been discussed for various values of R in Fig.4 which show that the spin or microrotations are similar for all cases of R.

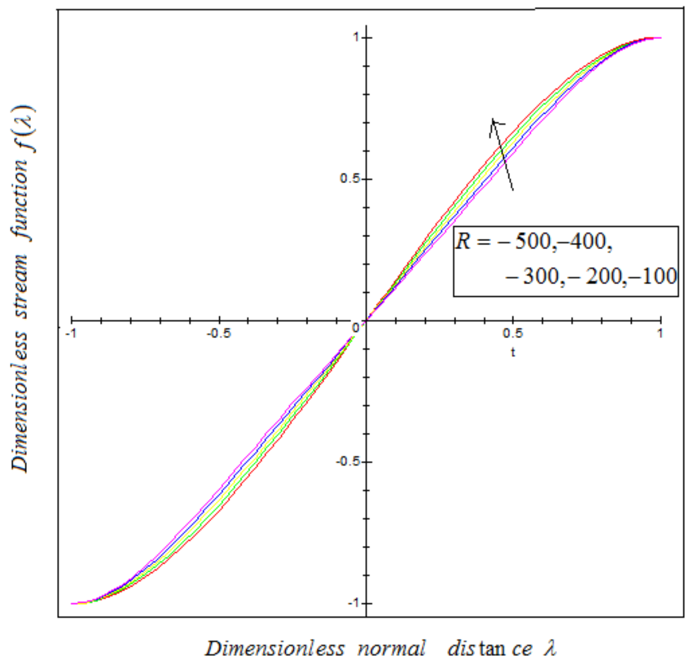

Fig. 2: Dimensionless stream function $f(\lambda)$ versus the dimensionless normal distance λ. The curves are plotted for the various values of the Reynolds number using the eqn. (19).

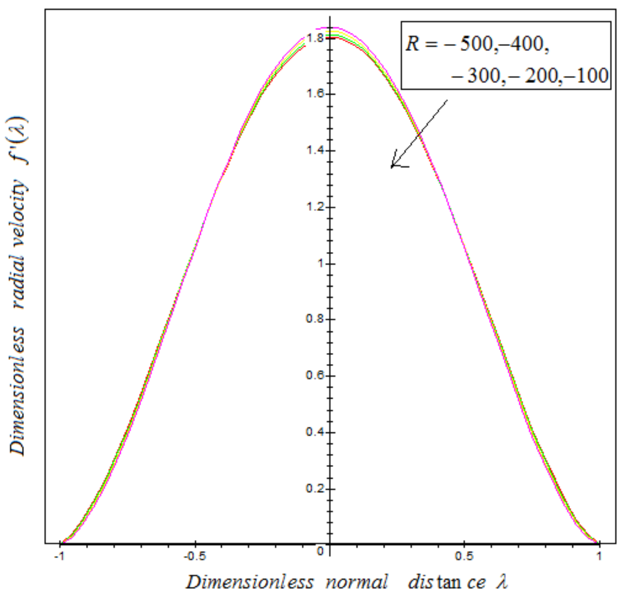

Fig. 3: Dimensionless radial velocity $f'(\lambda)$ versus the dimensionless normal distance λ. The curves are plotted for the various values of the Reynolds number using the eqn. (20).

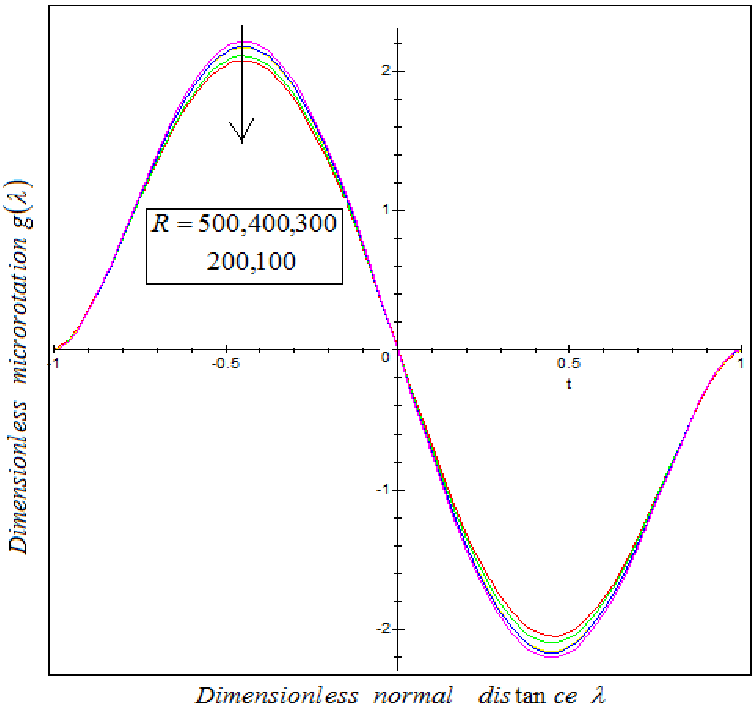

Fig. 4: Dimensionless microrotation $g(\lambda)$ versus the dimensionless normal distance λ. The curves are plotted for the various values of the Reynolds number using the eqn. (21).

5. Conclusion

In this paper we have derived the analytical expressions for the steady viscous flow of a micropolar fluid driven by injection between two porous disks with the Homotopy analysis method. The auxiliary parameter h provides us with a convenient way to adjust and control the convergence and its rate for the solutions series. We have obtained graphically the results for variations in the axial, radial velocities and the microrotation for different values of the Reynolds number R. Also we have computed the table values for different positions of the dimensionless normal distance λ.

Finally, it has been attempted to show the capabilities and wide-range applications of the Homotopy analysis method in comparison with the numerical solution of micropolar flow in a porous channel.

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Appendix: A

Basic concept of the Homotopy analysis method

Consider the following differential equation:

$$N[u(t)] = 0 \tag{A.1}$$

Where N is a nonlinear operator, t denotes an independent variable, $u(t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the conventional Homotopy method, Liao [33] constructed the so-called zero-order deformation equation as:

$$(1-p)L[\varphi(t;p) - u_0(t)] = phH(t)N[\varphi(t;p)] \tag{A.2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L an auxiliary linear operator, $u_0\ (t)$ is an initial guess of $u(t)$, $\varphi(t:p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary unknowns in HAM. Obviously, when $p = 0$ and $p = 1$, it holds:

$$\varphi(t;0) = u_0(t) \text{ and } \varphi(t;1) = u(t) \tag{A.3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(t;p)$ varies from the initial guess $u_0(t)$ to the solution $u\,(t)$. Expanding $\varphi(t;p)$ in Taylor series with respect to p, we have:

$$\varphi(t;p)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t)p^m \tag{A.4}$$

where

$$u_m(t)=\frac{1}{m!}\frac{\partial^m\varphi(t;p)}{\partial p^m}\Big|_{p=0} \tag{A.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (A.4) converges at p =1 then we have:

$$u(t)=u_0(t)+\sum_{m=1}^{+\infty}u_m(t). \tag{A.6}$$

Differentiating (A.2) for $m$times with respect to the embedding parameter p, and then setting p = 0 and finally dividing them by $m!$, we will have the so-called m th-order deformation equation as:

$$L[u_m-\chi_m u_{m-1}]=hH(t)\Re_m(\vec{u}_{m-1}) \tag{A.7}$$

where

$$\Re_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(t;p)]}{\partial p^{m-1}} \tag{A.8}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases} \tag{A.9}$$

Applying L^{-1} on both side of equation (A7), we get

$$u_m(t)=\chi_m u_{m-1}(t)+hL^{-1}[H(t)\Re_m(\vec{u}_{m-1})] \tag{A10}$$

In this way, it is easily to obtain u_m for $m\ge 1$, at M^{th} order, we have

$$u(t)=\sum_{m=0}^{M}u_m(t) \tag{A.11}$$

When $M\to+\infty$, we get an accurate approximation of the original equation (A.1). For the convergence of the above method we refer the reader to Liao [34]. If equation (A.1) admits unique solution, then this method will produce the unique solution.

Appendix: B

Solution of the non-linear differential eqns. (15), (16) and (18) using the Homotopy analysis method

This appendix contains the derivation of the analytical expressions eqns. (19), (20) and (21) for $f(\lambda)$ and $g(\lambda)$ using the Homotopy analysis method.

The eqns. (15), (16) and (18) are as follows,

$$f^{iv} - c_1 g'' - 2Rff''' = 0, \tag{B.1}$$

$$g'' + c_2\left(f'' - 2g\right) - c_3\left(f' g - 2fg'\right) = 0, \tag{B.2}$$

With the following boundary conditions

$$\begin{aligned} &f(1) = 1, && f(-1) = -1, \\ &f'(1) = 0, && f'(-1) = 0, \\ &g(1) = 0, && g(-1) = 0. \end{aligned} \tag{B.3}$$

We construct the Homotopy for these equations as follows,

$$(1-p)\left(f^{iv}\right) = h\,p\left(f^{iv} - c_1 g - 2Rff''\right) \tag{B.4}$$

$$(1-p)\left(g''\right) = h\,p\left(g'' + c_2\left(f'' - 2g\right) - c_3\left(f' g - 2f' g\right)\right) \tag{B.5}$$

The approximate solution for the eqns. (B.4) and (B.5) is given by,

$$f = f_0 + p\,f_1 + p^2 f_2 + \ldots.. \tag{B.6}$$

$$g = g_0 + p\,g_1 + p^2 g_2 + \ldots... \tag{B.7}$$

By substituting the eqn. (B.6) into the eqn. (B.4) we get,

$$(1-p)\left(\frac{d^4\left(f_0 + p\,f_1 + p^2 f_2 + \ldots.\right)}{d\lambda^4}\right) = hp\left(\begin{array}{l}\dfrac{d^4\left(f_0 + p\,f_1 + p^2 f_2 + \ldots.\right)}{d\lambda^4} - c_1\left(g_0 + pg_1 + p^2 g_2 + \ldots\right) \\ -2R\left(f_0 + pf_1 + p^2 f_2 + \ldots\right)\dfrac{d^2}{d\lambda^2}\left(f_0 + pf_1 + p^2 f_2 + \ldots.\right)\end{array}\right) \tag{B.8}$$

Similarly on substituting the eqn. (B.7) into an eqn. (B.5) we get,

$$(1-p)\left(\frac{d^2\left(g_0 + pg_1 + p^2 g_2 + \ldots.\right)}{d\lambda^2}\right) = hp\left(\begin{array}{l}\dfrac{d^2\left(g_0 + pg_1 + p^2 g_2 + \ldots.\right)}{d\lambda^2} + c_2\left(\begin{array}{l}\dfrac{d^2\left(f_0 + pf_1 + p^2 f_2 + \ldots.\right)}{d\lambda^2} \\ -2\left(g_0 + pg_1 + p^2 g_2 + \ldots\right)\end{array}\right) \\ -c_3\left(\begin{array}{l}\dfrac{d\ \left(f_0 + pf_1 + p^2 f_2 + \ldots.\right)}{d\lambda}\left(g_0 + pg_1 + p^2 g_2 + \ldots\right) \\ -2\left(\dfrac{d\left(f_0 + pf_1 + p^2 f_2 + ..\right)}{d\lambda}\right)\left(g_0 + pg_1 + p^2 g_2 + \ldots\right)\end{array}\right)\end{array}\right) \tag{B.9}$$

Now equating the coefficients of p^0 and p^1 in (B.8) we get the following equations,

$$p^0 : \frac{d^4 f_0}{d\lambda^4} = 0 \tag{B.10}$$

$$p^1 : \frac{d^4 f_1}{d\lambda^4} = c_1 \frac{d^2 g_0}{d\lambda^2} + 2R\, f_0 \frac{d^3 f_0}{d\lambda^3} \tag{B.11}$$

Similarly equating the coefficients of p^0 and p^1 in an eqn.(B.9) we obtain the following equations,

$$p^0 : \frac{d^2 g_0}{d\lambda^2} = 0 \tag{B.12}$$

$$p^1 : \frac{d^2 g_1}{d\lambda^2} + c_2 \frac{d^2 f_0}{d\lambda^2} = 0 \tag{B.13}$$

$$p^2 : \frac{d^2 g_2}{d\lambda^2} + c_2\left(\frac{d^2 f_1}{d\lambda^2} - 2g_1\right) - c_3\left(\frac{df_0}{d\lambda} g_1 - 2 f_0 \frac{dg_1}{d\lambda}\right) = 0 \tag{B.14}$$

The initial approximations are as follows:

$$f_0(1) = 1,\quad f_0(-1) = -1,\quad f_0^{'}(1) = 0,\quad f_0^{'}(-1) = 0 \tag{B.15}$$

$$f_1(1) = 0,\quad f_1(-1) = 0,\quad f_1^{'}(1) = 0,\quad f_1^{'}(-1) = 0 \tag{B.16}$$

$$g_0(1) = 0,\quad g_0(-1) = 0 \tag{B.17}$$

$$g_1(1) = 0,\quad g_1(-1) = 0 \tag{B.18}$$

$$g_2(1) = 0,\quad g_2(-1) = 0 \tag{B.19}$$

Solving the eqns. (B.10) and (B.15) we obtain the initial solution f_0 as,

$$f_0(\lambda) = -\frac{1}{2}\lambda^3 + \frac{3}{2}\lambda \tag{B.20}$$

And on solving the eqns. (B.11) and (B.16) we obtain the solution f_1 as,

$$f_1(\lambda) = -h\left(R\left(\frac{1}{280}\lambda^7 - \frac{3}{40}\lambda^5\right) + \frac{39}{280}R\lambda^3 - \frac{19}{280}R\lambda\right) \tag{B.21}$$

Similarly on solving the eqns. (B.12) and (B.17) we obtain the initial solution g_0 as,

$$g_0(\lambda) = 0 \tag{B.22}$$

On solving the eqns. (B.13) and (B.18) we obtain the initial solution g_1 as,

$$g_1(\lambda) = -h\left(\frac{1}{2}\lambda^3 - \frac{1}{2}\lambda\right) \tag{B.23}$$

And on solving the eqns. (B.14) and (B.19) we obtain the initial solution g_2 as,

$$g_2(\lambda) = h^2\left(\begin{array}{l} c_3\left(\frac{1}{28}\lambda^7 - \frac{3}{40}\lambda^6 - \frac{1}{4}\lambda^5 + \frac{1}{4}\lambda^4 + \frac{1}{4}\lambda^3 - \frac{3}{8}\lambda^2\right) \\ \qquad - c_2\left(R\left(\frac{1}{280}\lambda^7 - \frac{3}{40}\lambda^5\right) + \frac{39}{280}R\lambda^3 - \frac{1}{20}\lambda^5 + \frac{1}{6}\lambda^3\right) + A\lambda + B \end{array}\right) \tag{B.24}$$

Where

$A = c_2\left(\frac{19}{280}R + \frac{7}{60}\right) - \frac{1}{28}c_3$ and $B = \frac{1}{5}c_3$.

According to Homotopy analysis method we have

$$f = \lim_{p\to 1} f(\lambda) = f_0 + f_1 \tag{B.25}$$

$$g = \lim_{p\to 1} g(\lambda) = g_0 + g_1 + g_2 \tag{B.26}$$

Using the eqns. (B.20) and (B.21) in eqn. (B.25) and on substituting eqns. (B.22), (B.23) and (B.24) in (B.26) we get the solution as given in the text eqn. (19) and (21).

Appendix C:

Nomenclature

Symbols	Meaning
f	Dimensionless stream function
g	Dimensionless microrotation
j	Micro-inertia density
k	Coupling coefficient
P	Pressure
p	Embedding parameter
q	Mass transfer parameter
R	Reynolds number
s	Microrotation boundary condition
u	Velocity component in the x direction
x	Dimensional vertical coordinate
y	Dimensional horizontal coordinate
ρ	Fluid density
λ	Dimensionless normal distance
μ	Dynamic viscosity
v	Kinematic viscosity
v_s	Thermal diffusivity