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*Article*

# An Efficient Technique for Time-Fractional Wave Equations

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**Abstract:** In this paper, we consider Reduced Differential Transform Method (RDTM) for finding solution of multi-dimensional time -fractional wave equations. This paper presents a reliable technique in order to obtain the solutions for time-fractional wave equations with initial profile. The results reveal that the proposed technique is very effective, simple and straight forward and can be easily applied for other problems in mathematical physics.

**Keywords**: Time-fractional wave equations, reduced differential transform method, Maple.

**Mathematics Subject Classification:** 35Q79

## 1. Introduction

Partial Differential equations (PDEs) [1-18] emerge in many engineering and scientific mathematical modeling problems. In order to better understand these physical problems as well as further apply these physical phenomena in practical scientific research, it is important to find their exact solutions. The investigation of exact solution of these equations is interesting and extremely important. In the past several decades, many authors mainly had paid attention to study the solution of such equations by using various developed methods. Recently, the Variational Iteration Method (VIM) [1-7] has been successfully applied to handle various kinds of physical problems, for example, heat and wave-like equations [1], quadratic Riccati differential [2], Klein-Gordon equation [3], fractional differential equations [4], nonlinear differential equations [5], nonlinear thermo elasticity [6], nonlinear wave equations [7]. In Refs. [8-14] Adomian's Decomposition Method (ADM) [8-10], Homotopy Perturbation Method (HPM) [11], Homotopy Analysis Method (HAM) [12, 13] and Variation of Parameter Method (VPM) [14, 15] are successfully applied to obtain the exact solution of many

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physical models. In the present article, we use Reduced Differential Transform Method (RDTM) [16-18], to construct an appropriate solution to wave equations of fractional order.

The reduced differential transform technique is an iterative procedure for obtaining Taylor series solution of differential equations. This method reduces the size of computational work and easily applicable to many physical problems.

## 2. Definitions

The basic definitions of Reduced Differential Transform Method are explained as follows:

**Definition 2.1** If the function $u\ (x,t)$ is analytic and differentiated continuously with respect to $t$, then let

$$U_k\ (x) = \frac{1}{\Gamma(\mathrm{k}\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\ u(x,t)\right]_{t=0}, \tag{1}$$

where $\alpha$ is a parameter describing the order of time-fractional derivative and the $t-$dimensional spectrum function $U_k\ (x)$ is the transformed function. In this article, the lowercase $u\ (x,t)$ represents the original function and the uppercase $U_k\ (x)$ represents the transformed function.

**Table 1.** Reduced Differential Transform Method

| Functional form | Transformed form |
|---|---|
| $u\ (x,t)$ | $U_k\ (x) = \dfrac{1}{\Gamma(\mathrm{k}\alpha+1)} \left[\dfrac{\partial^{k\alpha}}{\partial t^{k\alpha}}\ u(x,t)\right]_{t=0}$ |
| $w\ (x,t) = u(x,t) \pm v(x,t)$ | $W_k\ (x) = U_k\ (x) \pm V_k\ (x)$ |
| $w\ (x,t) = \alpha\ u\ (x,t)$ | $W_k\ (x) = \alpha\ U_k\ (x)$ |
| $w\ (x,t) = u(x,t)v\ (x,t)$ | $W_k\ (x) = \sum_{n=0}^{k} U_n\ V_{k-n}\ (x) = \sum_{n=0}^{k} V_n\ U_{k-n}$ |
| $w\ (x,t) = \dfrac{\partial^n}{\partial y^n}\ u(x,t)$ | $W_k\ (x) = (k+1)(k+2)\ldots(k+n)U_{k+n}(x)$ |
| $w\ (x,t) = x^m y^n\ u(x,t)$ | $W_k\ (x) = x^m U_{k-n}\ (x)$ |
| $w\ (x,t) = \dfrac{\partial^{N\alpha}}{\partial t^{N\alpha}}\ u(x,t)$ | $W_k\ (x) = \dfrac{\Gamma(\mathrm{k}\alpha+\mathrm{N}\alpha+1)}{\Gamma(\mathrm{k}\alpha+1)} U_{k+N}\ (x)$ |

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**Definition 2.2** The differential inverse transform of $U_k\ (x)$ is defined as follows:

$$u\ (x,t) = \sum_{k=0}^{\infty} U_k\ (x) t^{k\alpha}. \tag{2}$$

Combining equation (1) and equation (2), it can be obtained that

$$u\ (x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\ u(x,t)\right]_{t=0} t^{k\alpha}\ . \tag{3}$$

From the above definitions, it can be understand that the concept of reduced differential transform method is derived from the power series expansion of a function. The mathematical operations performed by reduced differential transform method are listed in Table 1.

## 3 .Numerical Applications

In this section, we use Reduced Differential Transform Method (RDTM) to solve time-fractional wave equations. To show the reliability and efficiency of the proposed scheme, some examples are presented.

**Example 3.1** Consider the one dimensional homogeneous time-fractional wave equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial x^2},\quad 0 < x < \pi,\ t > 0,\ 0 < \alpha \le 1, \tag{4}$$

with the initial conditions $u(x,0) = \sin x\,, u_t(x,0) = 0.$ (5)

Applying the Reduced Differential Transform to equation (4), we obtain the following iteration relation,

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\ (x) = \left[\frac{\partial^2}{\partial x^2}\ U_k\ (x)\right]. \tag{6}$$

Where the $t$ −dimensional spectrum function $U_k\ (x)$ is the transform function.

From the initial conditions (5), we write,

$$U_0(x) = \ sinx, \quad U_1(x) = 0.$$

Using iteration equation (6), we obtain the following values 0f $U_k\ (x)$ successively,

$$U_2(x) = \frac{-sinx}{\Gamma(2\alpha+1)},\ U_3(x) = 0\,, U_4(x) = \frac{sinx}{\Gamma(4\alpha+1)}\,, U_5(x) = 0\,, U_6(x) = \frac{-sinx}{\Gamma(6\alpha+1)} \ldots$$

Finally the fractional differential inverse transform of $U_k\ (x)$ gives,

$$u\ (x,t) = \sum_{k=0}^{\infty} U_k\ (x) t^{k\alpha} = sinx - \frac{sinx}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{sinx}{\Gamma(4\alpha+1)} t^{4\alpha} - \frac{sinx}{\Gamma(6\alpha+1)} t^{6\alpha} \ldots$$

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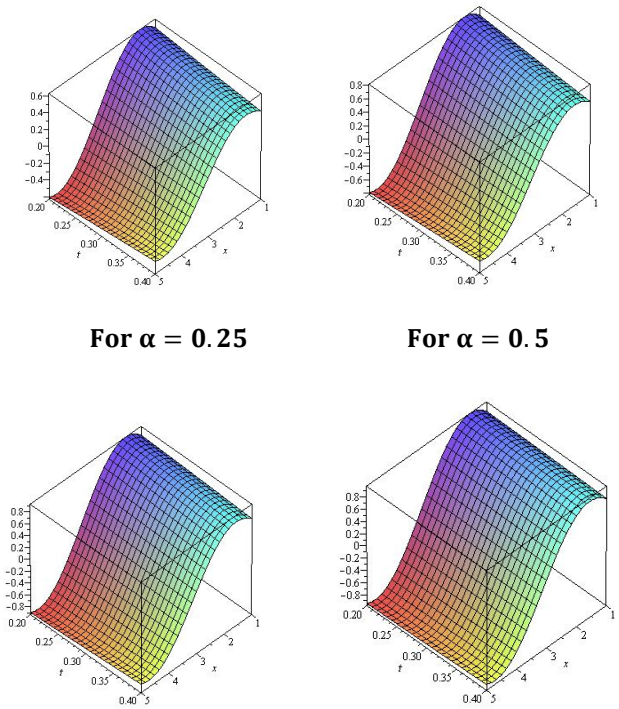

**For $\alpha = 0.25$** **For $\alpha = 0.5$**

**For $\alpha = 0.75$** **For $\alpha = 1$**

**Fig. 1**: 3D plot of $u(x,t)$ for $\alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.1

**Example 3.2** Consider the one dimensional inhomogeneous time -fractional wave equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial x^2} - 2\,, \quad 0 < x < \pi,\ t > 0,\ 0 < \alpha \leq 1, \tag{7}$$

with the initial conditions $u(x,0) = x^2,\ u_t(x,0) = sinx.$ (8)

Applying the Reduced Differential Transform to equation (7), we obtain the following iteration relation,

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\,(x) = \frac{\partial^2}{\partial x^2}\, U_k\,(x) - 2\delta\,(k). \tag{9}$$

Where the $t-$dimensional spectrum function $U_k$ (x) is the transform function.

From the initial conditions (8), we write,

$$U_0(x) = x^2\,, \qquad U_1(x) = sinx.$$

Using iteration equation (9), we obtain the following values 0f $U_k$ (x) successively,

$U_2(x) = 0,\ U_3(x) = -\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} sinx, U_4(x) = 0, U_5(x) = \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)} sinx, U_6(x) = 0, U_7(x) = \frac{\Gamma(\alpha+1)}{\Gamma(7\alpha+1)} sinx, \ldots$

Finally the fractional differential inverses transform of $U_k$ (x) gives us,

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$$u\,(x,t) = \sum_{k=0}^{\infty} U_k\,(x) t^{k\alpha}$$

$$= x^2 + sinx\; t^{\alpha} - \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} sinx\; t^{3\alpha} + \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)} sinx\; t^{5\alpha} - \frac{\Gamma(\alpha+1)}{\Gamma(7\alpha+1)} sinx\; t^{7\alpha} + ..$$

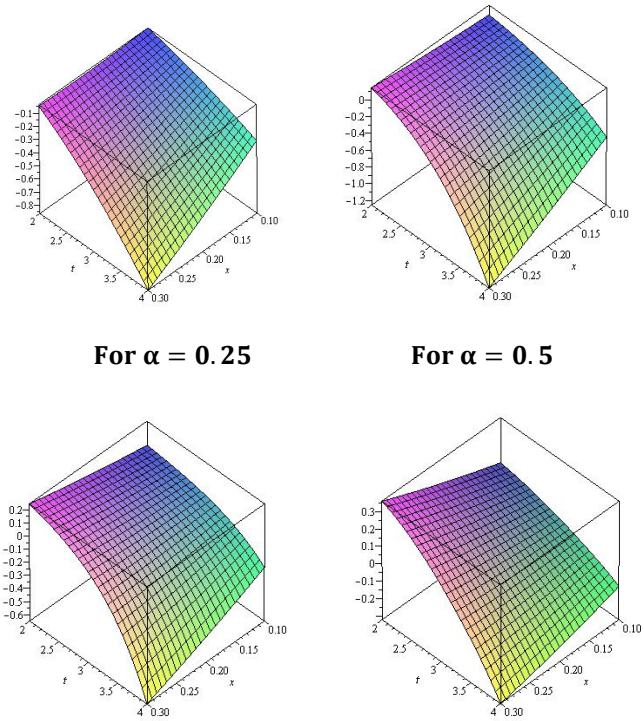


**Fig. 2:** 3D plot of $u(x,t)$ for $\alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.2

**Example 3.3** Consider the two dimensional homogeneous time fractional wave equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = 2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2}, \quad 0 < x, y < \pi, \; t > 0, \; 0 < \alpha \le 1, \tag{10}$$

with the initial conditions $u(x,y,0) = sinx\; siny, \;\; u_t(x,y,0) = 0.$ (11)

Applying the Reduced Differential Transform to equation (10), we obtain the following iteration relation,

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\,(x,y) = 2\frac{\partial^2}{\partial x^2}\, U_k\,(x,y) + 2\frac{\partial^2}{\partial y^2}\, U_k\,(x,y). \tag{12}$$

Where the $t-$dimensional spectrum function $U_k\,(x,y)$ is the transform function.

From the initial conditions (11), we write,

$$U_0(x,y) = \; sinx\; siny, \quad U_1(x,y) = 0.$$

Using iteration equation (12), we obtain the following values 0f $U_k\,(x,y)$ successively,

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$$U_2(x,y) = -\frac{4\ sinx\ siny}{\Gamma(2\alpha+1)}, U_3(x,y) = 0, U_4(x,y) = \frac{16\ sinx\ siny}{\Gamma(4\alpha+1)}, U_5(x,y) = 0, U_6(x,y) = -\frac{32\ sinx\ siny}{\Gamma(6\alpha+1)} \ldots$$

Finally the differential inverses transform of $U_k\ (x,y)$ gives us

$$u\ (x,y,t) = \sum_{k=0}^{\infty} U_k\ (x,y) t^{k\alpha}$$

$$= \ sinx\ siny - \frac{4\ sinx\ siny}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{16\ sinx\ siny}{\Gamma(4\alpha+1)} t^{4\alpha} - \frac{32\ sinx\ siny}{\Gamma(6\alpha+1)} t^{6\alpha} + \cdots$$

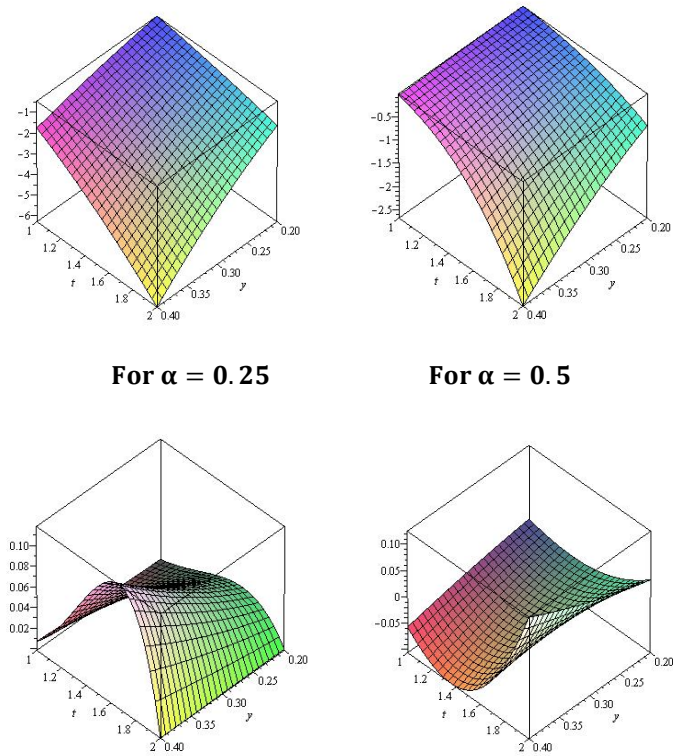


**Fig. 3:** 3D plot of $u(x,y,t)$ for $x = y, \alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.3

**Example 3.4** Consider the two dimensional inhomogeneous time fractional wave equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + cosy, \quad 0 < x, y < \pi, \ t > 0, \ 0 < \ \alpha \leq 1, \tag{13}$$

with the initial conditions $u(x,y,0) = cosy, \quad u_t(x,y,0) = sinx.$ (14)

Applying the Reduced Differential Transform to equation (13), we obtain the following iteration relation,

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\ (x,y) = \frac{\partial^2}{\partial x^2}\ U_k\ (x,y) + \frac{\partial^2}{\partial y^2}\ U_k\ (x,y) + \delta\ (k) cosy. \tag{15}$$

Where the $t-$dimensional spectrum function $U_k\ (x,y)$ is the transform function.

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From the initial conditions (14), we write,

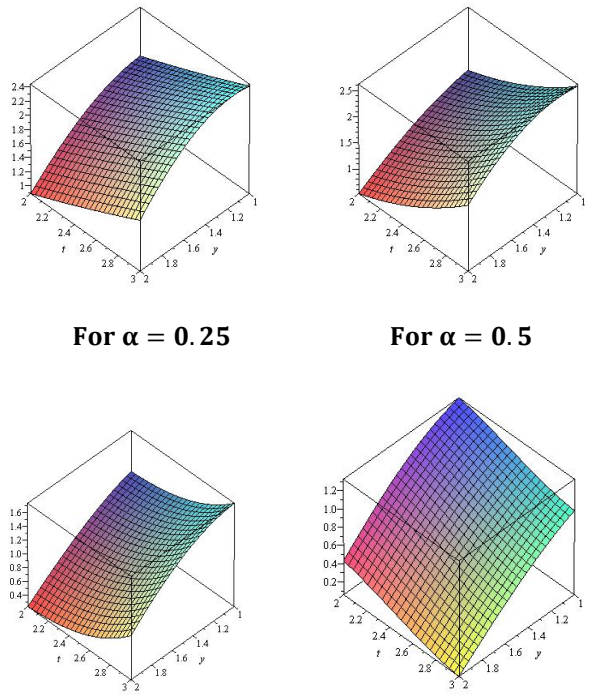
$$U_0(x,y) = cosy, \qquad U_1(x,y) = sinx.$$

Using iteration equation (15), we obtain the following values 0f $U_k\,(x,y)$ successively,

$$U_2(x,y) = 0, U_3(x,y) = -\frac{sinx\,\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}, U_4(x,y) = 0\,, U_5(x,y) = \frac{sinx\,\Gamma(\alpha+1)}{\Gamma(5\alpha+1)}, U_6(x,y) = 0, \ldots$$

Finally the differential inverses transform of $U_k\,(x,y)$ gives us

$$u\,(x,y,t) = \sum_{k=0}^{\infty} U_k\,(x,y)t^{k\alpha} = cosy + sinx\,t^{\alpha} - \frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)} sinx\,t^{3\alpha} + \frac{\Gamma(\alpha+1)}{\Gamma(5\alpha+1)} sinx\,t^{5\alpha} -, \ldots$$



**For $\alpha = 0.25$** **For $\alpha = 0.5$**

**For $\alpha = 0.75$** **For $\alpha = 1$**

**Fig. 4:** 3D plot of $u(x,y,t)$ for $x = y, \alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.4

**Example3.5** Consider the three dimensional homogeneous time fractional wave equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = 3\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial^2 u}{\partial y^2}, \quad 0 < x,y,z < \pi,\ t > 0,\ 0 < \alpha \le 1, \tag{16}$$

with the initial conditions $u(x,y,z,0) = 0$,

$$u_t(x,y,z,0) = 3sinx\ siny\ sinz. \tag{17}$$

Applying the Reduced Differential Transform to equation (16), we obtain the following iteration relation,

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\,(x,y,z) = 3\frac{\partial^2}{\partial x^2}\,U_k\,(x,y,z) + 3\frac{\partial^2}{\partial y^2}\,U_k\,(x,y,z) + 3\frac{\partial^2}{\partial z^2}\,U_k\,(x,y,z). \tag{18}$$

Where the $t-$dimensional spectrum function $U_k\,(x,y,z)$ is the transform function.

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From the initial conditions (17), we write,

$$U_0(x,y,z) = 0,\ \ U_1(x,y,z) = 3sinx\ siny\ sinz$$

Using iteration equation (18), we obtain the following values 0f $\mathrm{U_k}\ (\mathrm{x,y,z})$ successively,

$$U_2(x,y,z) = 0, U_3(x,y,z) = -\frac{27sinx\ siny\ sinz}{\Gamma(3\alpha+1)}\Gamma(\alpha+1), U_4(x,y,z) = 0,$$

$$U_5(x,y,z) = \frac{243sinx\ siny\ sinz}{\Gamma(5\alpha+1)}\Gamma(\alpha+1), U_6(x,y,z) = 0, \ldots$$

Finally the differential inverses transform of $U_k\ (x,y,z)$ gives us

$$u(x,y,z,t) = \sum_{k=0}^{\infty} U_k\ (x,y,z)t^{k\alpha}$$

$$= 3sinx\ sinysinz\ t^{\alpha} - \frac{27sinx\ siny\ sinz}{\Gamma(3\alpha+1)}\Gamma(\alpha+1)\ t^{3\alpha}$$

$$+\frac{243sinx\ siny\ sinz}{\Gamma(5\alpha+1)}\Gamma(\alpha+1)\ t^{5\alpha} - \cdots$$

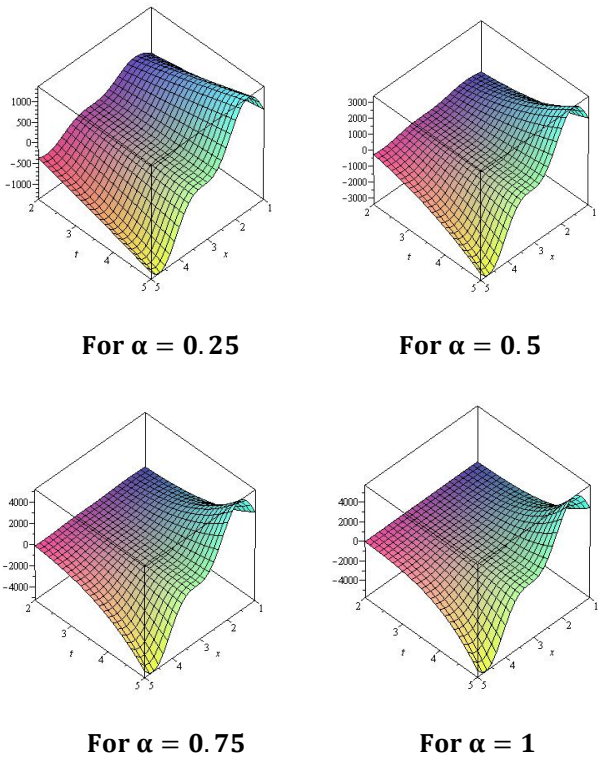


**Fig. 5:** 3D plot of $u(x,y,z,t)$ for $x = y = z, \alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.5

**Example3.6** Consider the three dimensional inhomogeneous time fractional wave equation

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} + sinx + siny,\ \ 0 < x,y,z < \pi,\ t > 0,\ 0 < \alpha \le 1, \tag{19}$$

with the initial conditions $u(x,y,z,0) = sinx + siny,\ u_t(x,y,z,0) = sinz.$ (20)

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Applying the Reduced Differential Transform to equation (19), we obtain the following iteration relation,

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\,(x,y,z) = \frac{\partial^2}{\partial x^2}\, U_k\,(x,y,z) + \frac{\partial^2}{\partial y^2}\, U_k\,(x,y,z) + \frac{\partial^2}{\partial z^2}\, U_k\,(x,y,z) + \delta(k) sinx + \delta(k) siny. \quad (21)$$

Where the $t-$dimensional spectrum function $U_k\,(x,y,z)$ is the transform function.

From the initial conditions (20), we write,

$$U_0(x,y,z) = sinx + siny,\ U_1(x,y,z) = \ sinz.$$

Using iteration equation (21), we obtain the following values 0f $U_k\,(x,y,z)$ successively,

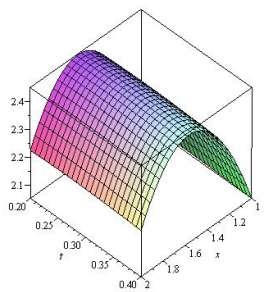
$$U_2(x,y,z) = 0, U_3(x,y,z) = -\frac{sinz}{\Gamma(3\alpha+1)}\Gamma(\alpha+1), U_4(x,y,z) = 0,$$

$$U_5(x,y,z) = \frac{sinz}{\Gamma(5\alpha+1)}\Gamma(\alpha+1), U_6(x,y,z) = 0, U_7(x,y,z) = -\frac{sinz}{\Gamma(7\alpha+1)}\Gamma(\alpha+1),..$$

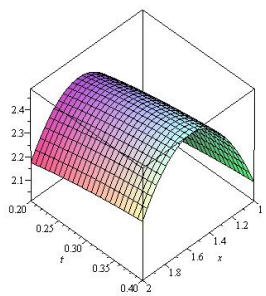
Finally the differential inverses transform of $U_k\,(x,y,z)$ gives us

$$u(x,y,z,t) = \sum_{k=0}^{\infty} U_k\,(x,y,z) t^{k\alpha}$$

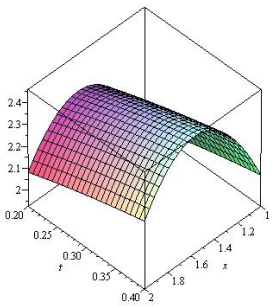
$$= sinx + siny + sinz\, t^{\alpha} - \frac{sinz}{\Gamma(3\alpha+1)}\Gamma(\alpha+1)\, t^{3\alpha} + \frac{sinz}{\Gamma(5\alpha+1)}\Gamma(\alpha+1)\, t^{5\alpha} - \cdots$$



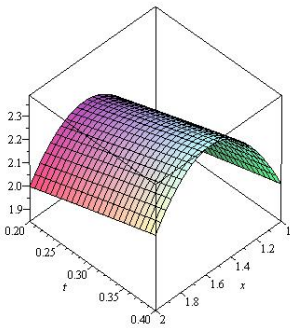
**For $\alpha = 0.25$**



**For $\alpha = 0.5$**



**For $\alpha = 0.75$**



**For $\alpha = 1$**

**Fig. 6:** 3D plot of $u(x,y,z,t)$ for $x = y = z, \alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.6

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**Example 3.7** Consider the following three dimensional non-linear time fractional wave like equation

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = \frac{\partial^2}{\partial x \partial y}\left(u_{xx}u_{yy}\right) - \frac{\partial^2}{\partial x \partial y}\left(xyu_x u_y\right) - u, \qquad 0 < \alpha \leq 1, \tag{22}$$

subject to the initial condition

$$u(x,y,0) = e^{xy}, u_t(x,y,0) = e^{xy}. \tag{23}$$

Applying the Reduced Differential Transform to equation (22), we obtain the following iteration relation

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}\,(x,y) = \frac{\partial^2}{\partial x \partial y}\left(U_{kxx}U_{kyy}\right) + \frac{\partial^2}{\partial x \partial y}\left(xyU_{kx}\,U_{ky}\right) - U_k. \tag{24}$$

Where the $t-$dimensional spectrum function $U_k\,(x,y)$ is the transform function.

From the initial conditions (23), we write, $u_0(x,y) = e^{xy}$, $u_1(x,y) = e^{xy}$

Now using iterative relation, we obtain the following values of $U_k(x)$ successively,

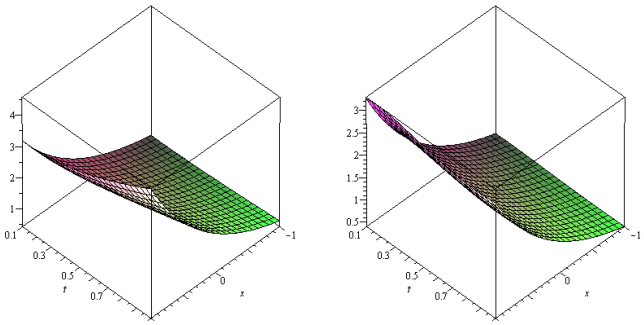
$$u_2(x,y) = -\frac{e^{xy}}{\Gamma(2\alpha+1)}, u_3(x,y) = -\frac{\Gamma(\alpha+1)e^{xy}}{\Gamma(3\alpha+1)}$$

$$u_4(x,y) = \frac{e^{xy}}{\Gamma(4\alpha+1)}, \quad u_5(x,y) = -\frac{\Gamma(\alpha+1)e^{xy}}{\Gamma(5\alpha+1)}, \ldots$$

Finally the reduced differential inverse transform of $U_k$(x,y) gives,

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^{\alpha k}.$$

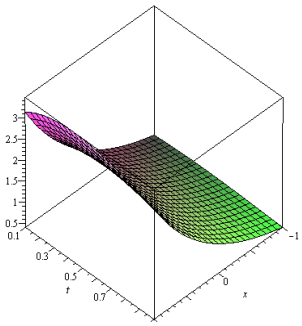
$$u\,(x,y,t) = e^{xy} + e^{xy}t^{\alpha} - \frac{e^{xy}}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{\Gamma(\alpha+1)e^{xy}}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{e^{xy}}{\Gamma(4\alpha+1)}t^{4\alpha} - \frac{\Gamma(\alpha+1)e^{xy}}{\Gamma(5\alpha+1)}t^{5\alpha}.$$



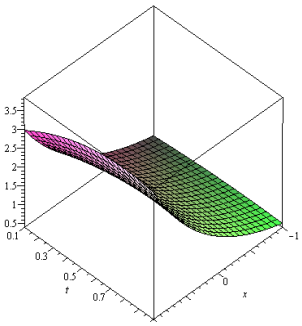
**For $\alpha = 0.25$**



**For $\alpha = 0.5$**



**For $\alpha = 0.75$**



**For $\alpha = 1$**

**Fig. 7:** 3D plot of $u(x,y,t)$ for $y = 1, \alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.7

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## 4. Conclusions

Reduced Differential Transform Method (RDTM) has been successfully applied to find appropriate solutions of multi-dimensional time fractional wave equations. It is observed that the proposed algorithm (RDTM) is fully compatible with the complexity of such problems. Numerical results explicitly reveal the complete reliability and efficiency of the proposed algorithm. In our work we use the Maple package to calculate the series obtained from reduced differential transform method and to show solution graphically.

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