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Article

Generalized Eulerian Integrals I

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Abstract: The present paper evaluated a new Eulerian integral associated with the product of two modified multivariable I-functions defined by Prasad [1], a generalized Lauricella function and the classes of multivariable polynomials with general arguments. The case concerning the Srivastava-Daoust polynomial [4] was studied as well and some remarks were given.

Keywords: Eulerian integral, modified multivariable H-function, generalized Lauricella function of several variables, generalized hypergeometric function, class of polynomials

Mathematics Subject Classification: 33C60, 82C31

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1. Introduction

In this paper, we consider a general class of Eulerian integral concerning the product of two multivariable I-functions, defined by Prasad [1], the generalized hypergeometric function and the classes of multivariable polynomials.

The generalized polynomials of multi-variables defined by Srivastava [3], is given in the following manner:

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}$$

$$A[N_1,K_1;\cdots;N_v,K_v] y_1^{K_1} \cdots y_v^{K_v} \tag{1.1}$$

where $\mathfrak{M}_1, \cdots, \mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots ; N_v, K_v]$ are arbitrary constants, real or complex. Srivastava and Garg [5] introduced and defined a general class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]=\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(L;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.2}$$

The coefficients are $B[L; R_1, \ldots, R_u]$ arbitrary constants, real or complex.

We shall note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v]$ and

$$b_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(L;R_1,\cdots,R_u)}{R_1!\cdots R_u!} \tag{1.3}$$

The multivariable I-function of r-variables defined by Prasad [1] generalized the multivariable function defined by Srivastava and Panda [6, 7]. It is defined in term of multiple Mellin-Barnes type integral:

$$I(z_1,z_2,\cdots,z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a^{(1)}_j,\alpha^{(1)}_j)_{1,p^{(1)}};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b^{(1)}_j,\beta^{(1)}_j)_{1,q^{(1)}};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right) \tag{1.4}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.5}$$

The defined integral of the above function, the existence and convergence conditions, see Y. N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.5) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

$$|arg z_i| < \frac{1}{2}\Omega_i\pi \,, \quad \text{where}$$

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r} \beta_{rk}^{(i)}\right) \quad (1.6)$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form:

$$I(z_1, \cdots, z_r) = 0(\ |z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\ ),\ max(\ |z_1|, \cdots, |z_r|\ ) \to 0$$

$$I(z_1, \cdots, z_r) = 0(\ |z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\ ),\ min(\ |z_1|, \cdots, |z_r|\ ) \to \infty$$

where $k = 1, \cdots, r : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m^{(k)}$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n^{(k)}$$

Consider a second multivariable I-function defined by Prasad [1]

$$I(z'_1, z'_2, ...z'_s) = I^{0,n'_2;0,n'_3;\cdots;0,n'_r:m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_s,q'_s:p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)}} \left(\begin{array}{c|l} z'_1 & (\mathrm{a}'_{2j};\alpha'^{(1)}_{2j},\alpha'^{(2)}_{2j})_{1,p'_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (\mathrm{b}'_{2j};\beta'^{(1)}_{2j},\beta'^{(2)}_{2j})_{1,q'_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{l} (\mathrm{a}'_{sj};\alpha'^{(1)}_{sj},\cdots,\alpha'_{sj}{}^{(s)})_{1,p'_s} : (a'^{(1)}_j,\alpha'^{(1)}_j)_{1,p'^{(1)}};\cdots;(a'_j{}^{(s)},\alpha'_j{}^{(s)})_{1,p'^{(s)}} \\ \\ (\mathrm{b}'_{sj};\beta'^{(1)}_{sj},\cdots,\beta'_{sj}{}^{(s)})_{1,q'_s} : (b'^{(1)}_j,\beta'^{(1)}_j)_{1,q'^{(1)}};\cdots;(b'_j{}^{(s)},\beta'_j{}^{(s)})_{1,q'^{(s)}} \end{array}\right) \quad (1.7)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\psi(t_1,\cdots,t_s)\prod_{i=1}^{s}\xi_i(t_i)z_i^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (1.8)$$

The defined integral of the above function, the existence and convergence conditions, see Y. N Prasad [1]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.5) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

where $|argz'_i| < \frac{1}{2}\Omega'_i\pi$,

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}} \alpha'^{(i)}_k - \sum_{k=n'^{(i)}+1}^{p'^{(i)}} \alpha'^{(i)}_k + \sum_{k=1}^{m'^{(i)}} \beta'^{(i)}_k - \sum_{k=m^{(i)}+1}^{q'^{(i)}} \beta'^{(i)}_k + \left(\sum_{k=1}^{n'_2} \alpha'^{(i)}_{2k} - \sum_{k=n_2+1}^{p'_2} \alpha'^{(i)}_{2k}\right)$$

$$+\cdots+\left(\sum_{k=1}^{n'_s} \alpha'^{(i)}_{sk} - \sum_{k=n'_s+1}^{p'_s} \alpha'^{(i)}_{sk}\right) - \left(\sum_{k=1}^{q'_2} \beta'^{(i)}_{2k} + \sum_{k=1}^{q'_3} \beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q'_s} \beta'^{(i)}_{sk}\right) \quad (1.9)$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form:

$$I(z'_1, \cdots, z'_s) = 0(\ |z'_1|^{\alpha'_1}, \cdots, |z'_s|^{\alpha'_s}\ ),\ max(\ |z'_1|, \cdots, |z'_s|\ ) \to 0$$

$$I(z'_1, \cdots, z'_s) = 0(\ |z'_1|^{\beta'_1}, \cdots, |z'_s|^{\beta'_s}\ ),\ min(\ |z'_1|, \cdots, |z'_s|\ ) \to \infty$$

where $k = 1, \cdots, z : \alpha''_k = min[Re(b'^{(k)}_j / \beta'^{(k)}_j)], j = 1, \cdots, m'^{(k)}$ and

$$\beta''_k = max[Re((a'^{(k)}_j - 1)/\alpha'^{(k)}_j)], j = 1, \cdots, n'^{(k)}$$

2. Integral Representation of Generalized Lauricella Function of Several Variables

In order to evaluate a number of integrals of multivariable I-functions, we first establish the formula:

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)\prod_{j=1}^{k}(af_j+g_j)^{\sigma_j}$$

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}\left(\begin{array}{c}(\alpha:h_1,\cdots,h_l,1,\cdots,1):(\lambda_1:1),\cdots,(\lambda_l:1);(-\sigma_1:1),\cdots,(-\sigma_k:1)\\ \cdots \\ (\alpha+\beta:h_1,\cdots,h_l,1,\cdots,1):-,\cdots,-;-,\cdots,-\end{array}\right.$$

$$\left.;\tau_1(b-a)^{h_1},\cdots,\tau_l(b-a)^{h_l},-\frac{(b-a)f_1}{af_1+g_1},\cdots,-\frac{(b-a)f_k}{af_k+g_k}\right) \quad (2.1)$$

where $a, b \in \mathbb{R}(a < b), \alpha, \beta, f_i, g_i, \sigma_i, \tau_j, h_j \in \mathbb{C}, \lambda_j \in \mathbb{R}^+(i = 1, \cdots, k; j = 1, \cdots, l)$

$$min(Re(\alpha), Re(\beta)) > 0, \max_{1 \leqslant j \leqslant l} \left\{ \left| \tau_j (b-a)^{h_j} \right| \right\} < 1, \max_{1 \leqslant j \leqslant k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1,$$

and $F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0}$ is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust [4, page. 454] given by :

$$F^{1:1,\cdots,1;1\cdots,1}_{1:0,\cdots,0;0,\cdots,0} \left(\begin{array}{c} (\alpha : h_1, \cdots, h_l, 1, \cdots, 1) : (\lambda_1 : 1), \cdots, (\lambda_l : 1); (-\sigma_1 : 1), \cdots, (-\sigma_k : 1) \\ \cdots \\ (\alpha + \beta : h_1, \cdots, h_l, 1, \cdots, 1) : -, \cdots, -; -, \cdots, - \end{array} \right.$$

$$; \tau_1 (b-a)^{h_1}, \cdots, \tau_l (b-a)^{h_l}, -\frac{(b-a)f_1}{af_1 + g_1}, \cdots, -\frac{(b-a)f_k}{af_k + g_k} \right) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \prod_{j=1}^{l} \Gamma(\lambda_j) \prod_{j=1}^{k} \Gamma(-\sigma_j)}$$

$$\frac{1}{(2\pi\omega)^{l+k}} \int_{L_1''} \cdots \int_{L_{l+k}''} \frac{\Gamma\left(\alpha + \sum_{j=1}^{l} h_j s_j + \sum_{j=1}^{k} s_{l+j}\right)}{\Gamma\left(\alpha + \beta + \sum_{j=1}^{l} h_j s_j + \sum_{j=1}^{k} s_{l+j}\right)} \prod_{j=1}^{l} \Gamma(\lambda_j + s_j) \prod_{j=1}^{k} \Gamma(-\sigma_j + s_{l+j})$$

$$\prod_{j=1}^{l+k} \Gamma(-s_j) z_1^{s_1} \cdots z_l^{s_l} z_{l+1}^{s_{l+1}} \cdots, z_{l+k}^{s_{l+k}} \mathrm{d}s_1 \cdots \mathrm{d}s_{l+k} \tag{2.2}$$

Here the contour $L_j''s$ are defined by $L_j'' = L_{\omega\zeta_j\infty}''(Re(\zeta_j) = v_j'')$ starting at the point $v_j'' - \omega\infty$ and terminating at the point $v_j'' + \omega\infty$ with $v_j'' \in \mathbb{R}(j = 1, \cdots, l)$ and each of the remaining contour $L_{l+1}'', \cdots, L_{l+k}''$ run from $-\omega\infty$ to $\omega\infty$.

(2.1) can be easily established by expanding $\prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\lambda_j}$ by means of the formula :

$$(1-z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r (|z| < 1) \tag{2.3}$$

Integrating term by term with the help of the integral given by Saigo and Saxena [2, page.93, eq. (3.2)] and applying the definition of the generalized Lauricella function [4, page.454].

3. Eulerian Integral

In this section, we note:

$$\theta_i = \prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j^{(i)}}, \zeta_j^{(i)} > 0 (i = 1, \cdots, r), \theta_i' = \prod_{j=1}^{l} \left[1 - \tau_j (t-a)^{h_i}\right]^{-\zeta_j'^{(i)}}, \zeta_j'^{(i)} > 0 (i = 1, \cdots, s)$$

$$\theta_i'' = \prod_{j=1}^{l} \left[1 - \tau_j(t-a)^{h_i}\right]^{-\zeta_j''^{(i)}}, \zeta_j''^{(i)} > 0 (i = 1, \cdots, u)$$

$$\theta_i''' = \prod_{j=1}^{l} \left[1 - \tau_j(t-a)^{h_i}\right]^{-\zeta_j'''^{(i)}}, \zeta_j'''^{(i)} > 0 (i = 1, \cdots, v) \tag{3.1}$$

$$U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; p_2', q_2'; p_3', q_3'; \cdots; p_{s-1}', q_{s-1}'; 0, 0; \cdots; 0, 0; 0, 0; \cdots; 0, 0 \tag{3.2}$$

$$V = 0, n_2; 0, n_3; \cdots; 0, n_{r-1}; 0, n_2'; 0, n_3'; \cdots; 0, n_{s-1}'; 0, 0; \cdots; 0, 0; 0, 0; \cdots; 0, 0 \tag{3.3}$$

$$X = m^{(1)}, n^{(1)}; \cdots; m^{(r)}, n^{(r)}; m'^{(1)}, n'^{(1)}; \cdots; m'^{(s)}, n'^{(s)}; 1, 0; \cdots; 1, 0; 1, 0; \cdots; 1, 0 \tag{3.4}$$

$$Y = p^{(1)}, q^{(1)}; \cdots; p^{(r)}, q^{(r)}; p'^{(1)}, q'^{(1)}; \cdots; p'^{(s)}, q'^{(s)}; 0, 1; \cdots; 0, 1; 0, 1; \cdots; 0, 1 \tag{3.5}$$

$$A = (a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)})_{1,p_2}; \cdots; (a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots, \alpha_{(r-1)k}^{(r-1)})_{1,p_{r-1}}; (a_{2k}'; \alpha_{2k}'^{(1)}, \alpha_{2k}'^{(2)})_{1,p_2'}; \cdots;$$

$$(a_{(s-1)k}'; \alpha_{(s-1)k}'^{(1)}, \alpha_{(s-1)k}'^{(2)}, \cdots, \alpha_{(s-1)k}'^{(s-1)})_{1,p_{s-1}'} \tag{3.6}$$

$$B = (b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)})_{1,q_2}; \cdots; (b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots, \beta_{(r-1)k}^{(r-1)})_{1,q_{r-1}}; (b_{2k}'; \beta_{2k}'^{(1)}, \beta_{2k}'^{(2)})_{1,q_2'}; \cdots;$$

$$(b_{(s-1)k}'; \beta_{(s-1)k}'^{(1)}, \beta_{(s-1)k}'^{(2)}, \cdots, \beta_{(s-1)k}'^{(s-1)})_{1,q_{s-1}'} \tag{3.7}$$

$$\mathbb{A} = (a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)}, 0, \cdots, 0, 0 \cdots, 0, 0, \cdots, 0)_{1,p_r};$$

$$(a_{sk}'; 0, \cdots, 0, \alpha_{sk}'^{(1)}, \alpha_{sk}'^{(2)}, \cdots, \alpha_{sk}'^{(s)}, 0, \cdots, 0, 0, \cdots, 0)_{1,p_s'} : (a_k^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}};$$

$$(a_k'^{(1)}, \alpha_k^{(1)})_{1,p^{(1)}}; \cdots; (a_k'^{(s)}, \alpha_k'^{(s)})_{1,p'^{(s)}}; (1, 0); \cdots; (1, 0); (1.0); \cdots; (1.0) \tag{3.8}$$

$$\mathbb{B} = (b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots, \beta_{rk}^{(r)}, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)_{1,q_r}$$

$$(b_{sk}'; 0, \cdots, 0, \beta_{sk}'^{(1)}, \beta_{sk}'^{(2)}, \cdots, \beta_{sk}'^{(s)}, 0, \cdots, 0, 0, \cdots, 0)_{1,q_s'} : (b_k^{(1)}, \beta_k^{(1)})_{1,q^{(1)}}; \cdots; (b_k^{(r)}, \beta_k^{(r)})_{1,q^{(r)}}$$

$$; (b_k'^{(1)}, \beta_k'^{(1)})_{1,q'^{(1)}}; \cdots; (b_k'^{(s)}, \beta_k'^{(s)})_{1,q'^{(s)}}; (0, 1); \cdots; (0, 1); (0, 1); \cdots; (0, 1) \tag{3.9}$$

$$A^* = (1 - \alpha - \sum_{i=1}^{u} R_i a_i - \sum_{i=1}^{v} K_i a_i'; \mu_1, \cdots, \mu_r, \mu_1', \cdots, \mu_s', h_1, \cdots, h_l, 1, \cdots, 1),$$

$$(1-\beta-\sum_{i=1}^{u}R_ib_i-\sum_{i=1}^{v}K_ib'_i;\rho_1,\cdots,\rho_r,\rho'_1,\cdots,\rho'_s,0,\cdots,0,0\cdots,0),$$

$$[1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j''^{(i)}-\sum_{i=1}^{v}K_i\zeta_j'''^{(i)};\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,\underset{\text{j}}{1},\cdots,0,0\cdots,0]_{1,l},$$

$$[1+\sigma_j-\sum_{i=1}^{u}R_i\lambda_j''^{(i)}-\sum_{i=1}^{v}K_i\lambda_j'''^{(i)};\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0\cdots,\underset{\text{j}}{1},\cdots,0]_{1,k} \tag{3.10}$$

$$B^*=(1-\alpha-\beta-\sum_{i=1}^{u}R_i(a_i+b_i)-\sum_{i=1}^{v}(a'_i+b'_i)K_i;\mu_1+\rho_1,\cdots,\mu_r+\rho_r,\mu'_1+\rho'_1,\cdots,\mu'_r+\rho'_r,$$

$$h_1,\cdots,h_l,1,\cdots,1),$$

$$[1-\lambda_j-\sum_{i=1}^{u}R_i\zeta_j''^{(i)}-\sum_{i=1}^{s}\zeta_j'''^{(i)}K_i;\zeta_j^{(1)},\cdots,\zeta_j^{(r)},\zeta_j'^{(1)}\cdots,\zeta_j'^{(s)},0,\cdots,0,0\cdots,0]_{1,l},$$

$$[1+\sigma_j-\sum_{i=1}^{u}R_i\lambda_j''^{(i)}-\sum_{i=1}^{v}\lambda_j'''^{(i)}K_i;\lambda_j^{(1)},\cdots,\lambda_j^{(r)},\lambda_j'^{(1)}\cdots,\lambda_j'^{(s)},0,\cdots,0,0,\cdots,0]_{1,k} \tag{3.11}$$

$$P_1=(b-a)^{\alpha+\beta-1}\left\{\prod_{j=1}^{h}(af_j+g_j)^{\sigma_j}\right\} \tag{3.12}$$

$$B_{u,v}=(b-a)^{\sum_{i=1}^{v}(a'_i+b'_i)K_i+\sum_{i=1}^{u}(a_i+b_i)R_i}\left\{\prod_{j=1}^{h}(af_j+g_j)^{-\sum_{i=1}^{v}\lambda_i'''K_i-\sum_{i=1}^{u}\lambda_i''R_i}\right\} \tag{3.13}$$

We have the general Eulerian integral.

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$S_L^{h_1,\cdots,h_u}\left(\begin{array}{c}\mathrm{z}_1''\theta_1''(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_u''\theta_u''(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}}\end{array}\right)$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} z_1'''\theta_1'''(t-a)^{a_1'}(b-t)^{b_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_v'''\theta_v'''(t-a)^{a_v'}(b-t)^{b_v'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^k(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t$$

$$= P_1 \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\prod_{k=1}^{u}z''^{R_k}\prod_{l=1}^{v}z'''^{K_l}\ a_vb_uB_{u,v}$$

$$I_{U:p_r+p_s'+l+k+2,q_r+q_s'+l+k+1;Y}^{V;0,n_r+n_s'+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{A}^*,\mathbb{A} \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^k(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \mathrm{B}\ ;\ \mathrm{B}^*,\mathbb{B} \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \tag{3.14}$$

We obtain the I-function of $r+s+k+l$ variables.

Provided that

(A) $a, b \in \mathbb{R}(a < b); \mu_i, \mu'_u, \rho_i, \rho'_u, \lambda_j^{(i)}, \lambda_j'^{(u)}, h_v \in \mathbb{R}^+$, $f_i, g_j, \tau_v, \sigma_j, \lambda_v \in \mathbb{C}$ $(i = 1, \cdots, r; j = 1, \cdots; k;$

$u = 1, \cdots, s; v = 1, \cdots, l), a_i, b_i, \lambda_j''^{(i)}, \zeta_j''^{(i)} \in \mathbb{R}^+, (i = 1, \cdots, u; j = 1, \cdots, k)$

$a'_i, b'_i, \lambda_j'''^{(i)}, \zeta_j'''^{(i)} \in \mathbb{R}^+, (i = 1, \cdots, v; j = 1, \cdots, k)$

(B) $a_{ij}, b_{ik}, \in \mathbb{C}$ $(i = 1, \cdots, r; j = 1, \cdots, p_i; k = 1, \cdots, q_i)$; $a_j^{(i)}, b_j^{(k)} \in \mathbb{C}$

$(i = 1, \cdots, r; j = 1, \cdots, p^{(i)}; k = 1, \cdots, q^{(i)})$

$a'_{ij}, b'_{ik}, \in \mathbb{C}$ $(i = 1, \cdots, s; j = 1, \cdots, p'_i; k = 1, \cdots, q'_i)$; $a_j'^{(i)}, b_j'^{(k)}, \in \mathbb{C}$

$(i = 1, \cdots, r; j = 1, \cdots, p'^{i)}; k = 1, \cdots, q'^{(i)})$

$\alpha_{ij}^{(k)}, \beta_{ij}^{(k)} \in \mathbb{R}^+$ ($(i = 1, \cdots, r, j = 1, \cdots, p_i, k = 1, \cdots, r)$; $\alpha_j^{(i)}, \beta_i^{(i)} \in \mathbb{R}^+$ $(i = 1, \cdots, r; j = 1, \cdots, p_i)$

$\alpha_{ij}'^{(k)}, \beta_{ij}'^{(k)} \in \mathbb{R}^+$ ($(i = 1, \cdots, s, j = 1, \cdots, p'_i, k = 1, \cdots, s)$; $\alpha_j'^{(i)}, \beta_i'^{(i)} \in \mathbb{R}^+$ $(i = 1, \cdots, s; j = 1, \cdots, p'_i)$

(C) $\max_{1 \leqslant j \leqslant k} \left\{ \left| \frac{(b-a)f_i}{af_i + g_i} \right| \right\} < 1, \ \max_{1 \leqslant j \leqslant l} \left\{ \left| \tau_j (b-a)^{h_j} \right| \right\} < 1$

(D) $$Re\left[\alpha + \sum_{j=1}^{r} \mu_j \min_{1 \leqslant k \leqslant m^{(i)}} \frac{b_k^{(j)}}{\beta_k^{(j)}} + \sum_{j=1}^{s} \mu'_i \min_{1 \leqslant k \leqslant m'^{(i)}} \frac{b_k'^{(j)}}{\beta_k'^{(j)}}\right] > 0$$

$$Re\left[\beta + \sum_{j=1}^{r} \rho_j \min_{1 \leqslant k \leqslant m^{(i)}} \frac{b_k^{(j)}}{\beta_k^{(j)}} + \sum_{j=1}^{s} \rho'_j \min_{1 \leqslant k \leqslant m'^{(i)}} \frac{b_k'^{(j)}}{\beta_k'^{(j)}}\right] > 0$$

(E) $$Re\left(\alpha + \sum_{i=1}^{v} K_i a'_i + \sum_{i=1}^{u} R_i a_i + \sum_{i=1}^{r} \mu_i s_i + \sum_{i=1}^{s} t_i \mu'_i\right) > 0$$

$$Re\left(\beta + \sum_{i=1}^{v} K_i b'_i + \sum_{i=1}^{u} R_i b_i + \sum_{i=1}^{r} v_i s_i + \sum_{i=1}^{s} t_i \rho'_i\right) > 0$$

$$Re\left(\lambda_j + \sum_{i=1}^{v} K_i \lambda_j'''^{(i)} + \sum_{i=1}^{u} R_i \lambda_j''^{(i)} + \sum_{i=1}^{r} s_i \zeta_j^{(i)} + \sum_{i=1}^{s} t_i \zeta_j'^{(i)}\right) > 0 (j = 1, \cdots, l);$$

$$Re\left(-\sigma_j+\sum_{i=1}^{v}K_i\lambda'''^{(i)}+\sum_{i=1}^{u}R_i\lambda_j''^{(i)}+\sum_{i=1}^{r}s_i\lambda_j^{(i)}+\sum_{i=1}^{s}t_i\lambda_j'^{(i)}\right)>0(j=1,\cdots,k);$$

(F) $$\Omega_i=\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+$$

$$\left(\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\right)-\mu_i-\rho_i$$

$$-\sum_{l=1}^{k}\lambda_j^{(i)}-\sum_{l=1}^{l}\zeta_j^{(i)}>0\quad(i=1,\cdots,r)$$

$$\Omega_i'=\sum_{k=1}^{n'^{(i)}}\alpha_k'^{(i)}-\sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha_k'^{(i)}+\sum_{k=1}^{m'^{(i)}}\beta_k'^{(i)}-\sum_{k=m^{(i)}+1}^{q'^{(i)}}\beta_k'^{(i)}+\left(\sum_{k=1}^{n_2'}\alpha_{2k}'^{(i)}-\sum_{k=n_2+1}^{p_2'}\alpha_{2k}'^{(i)}\right)+$$

$$\cdots+\left(\sum_{k=1}^{n_s'}\alpha_{sk}'^{(i)}-\sum_{k=n_s'+1}^{p_s'}\alpha_{sk}'^{(i)}\right)-\left(\sum_{k=1}^{q_2'}\beta_{2k}'^{(i)}+\sum_{k=1}^{q_3'}\beta_{3k}'^{(i)}+\cdots+\sum_{k=1}^{q_s'}\beta_{sk}'^{(i)}\right)-\mu_i'-\rho_i'$$

$$-\sum_{l=1}^{k}\lambda_j'^{(i)}-\sum_{l=1}^{l}\zeta_j'^{(i)}>0\quad(i=1,\cdots,s)$$

(G) $$\left|arg\left(z_i\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\zeta_j^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(i)}}\right)\right|<\frac{1}{2}\Omega_i\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,r)$$

$$\left|arg\left(z_i'\prod_{j=1}^{l}\left[1-\tau_j'(t-a)^{h_i'}\right]^{-\zeta_j'^{(i)}}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(i)}}\right)\right|<\frac{1}{2}\Omega_i'\pi\quad(a\leqslant t\leqslant b;i=1,\cdots,s)$$

Proof

To prove (3.14), first, we express in serie the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ with the help of (1.1), the class of multivariable polynomials $S_L^{h_1,\cdots,h_u}[.]$ in serie with the help of (1.2) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the I-functions of r-variables and s-variables defined by Prasad [1] in terms of Mellin-Barnes type contour integral with the help of (1.9) and (1.12) respectively and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process . Now collect the power of $\left[1-\tau_j(t-a)^{h_i}\right]$ with $(i=1,\cdots,r;j=1,\cdots,l)$ and collect the power of (f_jt+g_j) with $j=1,\cdots,k$. Use the equations (2.1) and (2.2) and express the result in Mellin-Barnes contour integral. Interpreting the $(r+s+k+l)$ dimensional Mellin-Barnes integral in multivariable I-function defined by Prasad [1], we obtain the equation (3.14).

Remarks

If a) $\rho_1 = \cdots, \rho_r = \rho'_1 = \cdots, \rho'_s = 0$; b) $\mu_1 = \cdots, \mu_r = \mu'_1 = \cdots, \mu'_s = 0$; c) We replace The multivariable I-functions by the multivariable H-functions defined by Srivastava and Panda [6,7] , we obtain the similar corresponding formulae.

4. Srivastava-Daoust Polynomial

b) If $$B(L;R_1,\cdots,R_u) = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta'_j+\cdots+R_u\theta_j^{(u)}}\prod_{j=1}^{B'}(b'_j)_{R_1\phi'_j}\cdots\prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi'_j+\cdots+R_u\psi_j^{(u)}}\prod_{j=1}^{D'}(d'_j)_{R_1\delta'_j}\cdots\prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}} \tag{4.1}$$

then the general class of multivariable polynomial $S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u]$ reduces to generalized Lauricella function defined by Srivastava et al [4].

$$F_{\bar{C}:D^{(1)};\cdots;D^{(u)}}^{1+\bar{A}:B^{(1)};\cdots;B^{(u)}}\left(\begin{array}{c|c} z_1 & (\text{-L}:R_1,\cdots,R_u),[(a);\theta',\cdots,\theta^{(u)}]:[(b');\phi'];\cdots;[(b^{()});\phi^{(u)}] \\ \cdot & \cdot \\ \cdot & [(c);\psi',\cdots,\psi^{(u)}]:[(d');\delta'];\cdots;[(d^{(u)});\delta^{(u)}] \\ z_u & \end{array}\right) \tag{4.2}$$

We have the following integral.

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\prod_{j=1}^{l}\left[1-\tau_j(t-a)^{h_i}\right]^{-\lambda_j}\prod_{j=1}^{k}(f_jt+g_j)^{\sigma_j}$$

$$F_{\bar{C}:D';\cdots;D^{(u)}}^{1+\bar{A}:B';\cdots;B^{(u)}}\left(\begin{array}{c} z''_1\theta''_1(t-a)^{a_1}(b-t)^{b_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z''_u\theta''_u(t-a)^{a_u}(b-t)^{b_u}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j''^{(u)}} \end{array}\right)$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} z'''_1\theta'''_1(t-a)^{a'_1}(b-t)^{b'_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z'''_v\theta'''_v(t-a)^{a'_v}(b-t)^{b'_v}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'''^{(v)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1\theta_1(t-a)^{\mu_1}(b-t)^{\rho_1}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r\theta_r(t-a)^{\mu_r}(b-t)^{\rho_r}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j^{(r)}} \end{array}\right)$$

$$I\left(\begin{array}{c} z_1'\theta_1'(t-a)^{\mu_1'}(b-t)^{\rho_1'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_s'\theta_s'(t-a)^{\mu_s'}(b-t)^{\rho_s'}\prod_{j=1}^{k}(f_jt+g_j)^{-\lambda_j'^{(s)}} \end{array}\right)\mathrm{d}t$$

$$= P_1 \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L} \prod_{k=1}^{u} z''^{R_k} \prod_{l=1}^{v} z'''^{K_l} \; a_v b_u B_{u,v}$$

$$I_{U:p_r+p_s'+l+k+2,q_r+q_s'+l+k+1;Y}^{V;0,n_r+n_s'+l+k+2;X}\left(\begin{array}{c|c} \frac{z_1(b-a)^{\mu_1+\rho_1}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(1)}}} & \mathrm{A}\ ;\ \mathrm{A}^*, \mathbb{A} \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_r(b-a)^{\mu_r+\rho_r}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j^{(r)}}} & \cdot \\ \frac{z_1'(b-a)^{\mu_1'+\rho_1'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(1)}}} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \frac{z_s'(b-a)^{\mu_s'+\rho_s'}}{\prod_{j=1}^{k}(af_j+g_j)^{\lambda_j'^{(s)}}} & \cdot \\ \tau_1(b-a)^{h_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ \tau_l(b-a)^{h_l} & \cdot \\ \frac{(b-a)f_1}{af_1+g_1} & \cdot \\ \cdots & \mathrm{B}\ ;\ \mathrm{B}^*, \mathbb{B} \\ \cdots & \\ \frac{(b-a)f_k}{af_k+g_k} & \end{array}\right) \quad \mathbf{(4.3)}$$

under the same conditions and notations that (3.14)

where $b_u = \dfrac{(-L)_{h_1R_1+\cdots+h_uR_u}B(E;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[E;R_1,\ldots,R_v]$ is defined by (4.1)

Remark:

By the following similar procedure, the results of this document can be extended to product of any finite number of multivariable I-functions defined by Prasad [1], classes of multivariable polynomials defined by Srivastava and Garg [5] and class of multivariable polynomials defined by Srivastava [3].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable I-functions defined by Prasad [1], a class of multivariable polynomials defined by Srivastava and Garg [5] and a class of multivariable polynomials defined by Srivastava [3] with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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