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Article

Existence of Solutions of Nonlinear Fractional Integrodifferential Equation with Analytic Semigroup

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Abstract: In this paper, we prove the existence and uniqueness of local mild and classical solutions of a class of nonlinear fractional integrodifferential equations in Banach space with analytic semigroup. Gelfand-Shilov principle and fractional powers of operators are used to establish the results.

Keywords: Fractional integrodifferential equation; Fractional powers; Mild and Classical solution; Analytic semigroup.

1. Introduction

In this paper, we are concerned with the following fractional integrodifferential equation in a Banach space X:

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f\big(t, u(t)\big) + \int_0^t h\left(t, s, u(s), \int_0^s k\big(s, \tau, u(\tau)\big) d\tau\right) ds, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

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where $0 < \alpha \leq 1, t > 0$. Let $J = [0, a]$ and let $-A$ is the infinitesimal generator of an analytic semigroup $Q(t), t \geq 0$. Let $f: J \times X \to X, h: J \times J \times X \times X \to X, k: J \times J \times X \to X$ be given nonlinear operators.

This type of research has been considered in Balachandran and Chandrasekaran [1], when the equation (1) is given with conventional (Classical) derivatives, also as many works [2, 3, 4, 5] and references cited therein. Fractional derivatives have been extensively applied in many fields, for example in Probability, Viscoelasticity, Electronics, Economics, mechanics as well as Biology. In recent years, they have been an object of investigations with much increasing interest. For more details on this theory and application, we refer the monographs of Lakshmikantham et al. [6], Miler and Ross [7], Podlubny [8] and the papers of [9, 10, 11, 12].

In this paper, we generalize the results of [1, 20]. The rest of this paper is organized as follows. In section 2, we give some preliminaries. In section 3, we prove our main theorem for (1) – (2).

2. Preliminaries

Here we assume that $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator in a Banach space X. Hence for convenience, we suppose that $\|Q(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$. We define the fractional power A^{-q} by

$$A^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty (t)^{q-1} Q(t) dt, q > 0.$$

For $0 < q \leq 1, A^q$ is a closed linear invertible operator with domain $D(A^q) \supset D(A)$ is dense in X. The closedness of A^q implies that $D(A^q)$, endowed with the graph norm of $A^q, \|u\|_{D(A)} = \|u\| + \|A^q u\|, u \in D(A^q)$, is a Banach space. Since A^q is invertible, and its graph norm $|\|.\||$ is equivalent to the norm $\|u\|_q = \|A^q u\|$. Thus $D(A^q)$ equipped with the norm$\|.\|_q$, is a Banach space, which we denote by X_q. Take $J = [0, a]$.

Following Gelfand and Shilov [13], we define fractional integral of order $\alpha > 0$ as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

also, the fractional derivative of the function f of order $0 < \alpha < 1$ as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-\alpha} ds,$$

where f is an abstract continuous function on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see [14].

Definition 1: By a solution of (1) – (2), we mean a function u with values in X such that:

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(1) u is continuous function on J and $u(t) \in D(A)$,

(2) $\frac{d^\alpha u}{dt^\alpha}$ exists and continuous on $(0, a), 0 < \alpha < 1$, and u satisfies (1) on $(0, a)$ and the initial condition (2).

Using Gelfand-Shilov principle [13], it is suitable to rewrite equation (1), (2) in the form

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f\big(s, u(s)\big) - Au(s) + \int_0^s h\left(s, \tau, u(\tau), \int_0^\tau k\big(\tau, \mu, u(\mu)\big) d\mu \right) d\tau \right] ds \quad (3)$$

Remark 1: Let us take in the considered problem, the inhomogeneous part is equal to an abstract continuous function $F(t)$, then from $(1) - (2)$, we have

$$D_t^\alpha u(t) + Au(t) = F(t), \quad (4)$$

$$u(0) = u_0. \quad (5)$$

According to [14, 15-19], we first apply the fractional integral on both sides of (4) and then using (5), we apply the Laplace transform on the new integral equations by considering a suitable one-sided stable probability density whose Laplace transform is given. Hence we can conclude that a solution of the problem $(4) - (5)$ can be formally represented by

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 \, d\theta + \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta) F(s) d\theta ds, \quad (6)$$

where

$$F(t) = f\big(t, u(t)\big) + \int_0^t h\left(t, s, u(s), \int_0^s k\big(s, \tau, u(\tau)\big) d\tau \right) ds$$

and ζ_α is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} \zeta_\alpha(\theta) d\theta = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1 + \alpha j)}, 0 < \alpha \le 1, x > 0.$$

3. Existence Theorem

To prove our main result we state the following lemma:

Lemma 1: Let $-A$ be the infinitesimal generator of an analytic semigroup $Q(t)$. If $0 \in \rho(A)$ then

(a) $Q(t): X \to D(A^q)$ for every $t > 0$ and $q \ge 0$

(b) For every $u \in D(A^q)$, we have $Q(t) A^q u = A^q Q(t) u$

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(c) For every $t > 0$ the operator $A^q Q(t)$ is bounded and $\|A^q Q(t)\| \leq M_q t^{-q}$.

For more details, see [21, section 2.6]

Theorem 1: Assume that

(i) $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $Q(t), t > 0$, in X.

(ii) For $q \geq 0$, the fractional power A^q satisfies $\|A^q Q(t)\| \leq M_q t^{-q}$ for $t > 0$, where M_q is a real constant.

(iii) $0 \in \rho(-A)$, the resolvent set.

(iv) For an open subset D of $J \times X_q, f: D \to X$ satisfies the condition, if for every $(t, u) \in D$ there is a neighborhood $V \subset D$ and constants $L \geq 0, 0 < \vartheta \leq 1$, such that

$$\|f(t_1, u_1) - f(t_2, u_2)\| \leq L\left(|t_1 - t_2|^{\vartheta} + \|u_1 - u_2\|_q\right) \tag{7}$$

for all $(t_i, u_i) \in V, i = 1,2$.

(v) For an open subset E of $J \times J \times X_q \times X_q, h: E \to X$ satisfies the condition, if for every $(t, s, u, v) \in E$ there is a neighborhood $U \subset E$ and constants $L_1 \geq 0, 0 < \vartheta \leq 1$, such that

$$\|h(t_1, s_1, u_1, v_1) - h(t_2, s_2, u_2, v_2)\|$$

$$\leq L_1\left(|t_1 - t_2|^{\vartheta} + |s_1 - s_2|^{\vartheta} + \|u_1 - u_2\|_q + \|v_1 - v_2\|_q\right) \tag{8}$$

for all $(t_i, s_i, u_i, v_i) \in U, i = 1,2$.

(vi) For an open subset P of $J \times J \times X_q, k: P \to X$ satisfies the condition, if for every $(t, s, u) \in P$ there is a neighborhood $W \subset P$ and constants $L_2 \geq 0, 0 < \vartheta \leq 1$, such that

$$\|k(t_1, s_1, u_1) - k(t_2, s_2, u_2)\| \leq L_2\left(|t_1 - t_2|^{\vartheta} + |s_1 - s_2|^{\vartheta} + \|u_1 - u_2\|_q\right) \tag{9}$$

for all $(t_i, s_i, u_i) \in W, i = 1,2$.

Then the Cauchy problem (1) – (2) has a unique local solution $u \in C([0, a): X) \cap C^1\big((0, a): X\big)$.

Proof: Choose $t^* > 0$ and $\delta > 0$ such that estimates (7) – (9) hold on the sets

$$V = \{(t, u): 0 \leq t \leq t^*, \|u - u_0\| \leq \delta\},$$

$$U = \{(t, s, u, v): 0 \leq t, s \leq t^*, \|u - u_0\| \leq \delta, \|v - v_0\| \leq \delta\},$$

and $W = \{(t, s, u): 0 \leq t, s \leq t^*, \|u - u_0\| \leq \delta\}$, respectively.

Let

$$B = \max_{0 \leq t < a} \|f(t, u_0)\|$$

and

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$$H = \max_{0 \le t, s \le t^*} \left\| h\left(t, s, u_0, \int_0^s k(s, \tau, u_0)\, d\tau \right) \right\|$$

and choose a such that for $0 \le t < a$,

$$\|Q(t^\alpha \theta) - I\| \|A^q u_0\| \le \frac{\delta}{2} \tag{10}$$

and

$$0 < a < \min\left\{ t^*, \left[\frac{\delta}{2} M_q^{-1} (1-q)(L\delta + B + L_1 \delta a + L_1 L_2 \delta a^2 + Ha)^{-1} \right]^{\frac{1}{\alpha(1-q)}} \right\}. \tag{11}$$

Let Y be a Banach space $C\big((0, a]: X\big)$ with usual supremum norm which we denote by $\|.\|_Y$. Define a map $F: Y \to Y$ by

$$Fy(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) A^q u_0\, d\theta$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-s)^\alpha \theta) \left[f\big(s, A^{-q} y(s)\big) \right.$$

$$\left. + \int_0^s h\left(s, \tau, A^{-q} y(\tau), \int_0^\tau k\big(\tau, \mu, A^{-q} y(\mu)\big) d\mu \right) d\tau \right] d\theta ds. \tag{12}$$

Since $\int_0^\infty \zeta_\alpha(\theta) d\theta = 1$, for every $y \in Y, FY(0) = A^q u_0$, Let S be the nonempty closed and bounded subset of Y defined by

$$S = \{y : y \in Y, y(0) = A^q u_0, \|y(t) - A^q u_0\| \le \delta\}.$$

For $y \in S$, we have

$$\|Fy(t) - A^q u_0\| \le \left\| Q(t^\alpha \theta) - I \right\| \left\| A^q u_0 \right\| +$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) \left\| A^q Q((t-s)^\alpha \theta) \right\| \left\| f\big(s, A^{-q} y(s)\big) - f(s, u_0) \right\| d\theta ds$$

$$+ \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) \left\| A^q Q((t-s)^\alpha \theta) \right\| \|f(s, u_0)\| d\theta ds$$

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$$+\alpha\int_0^t\int_0^\infty \theta\,(t-s)^{\alpha-1}\zeta_\alpha(\theta)\left\|A^qQ((t-s)^\alpha\theta)\right\|$$

$$\left\|\left[\int_0^s h\left(s,\tau,A^{-q}y(\tau),\int_0^\tau k\big(\tau,\mu,A^{-q}y(\mu)\big)d\mu\right)d\tau-\int_0^s h\left(s,\tau,u_0,\int_0^\tau k(\tau,\mu,u_0)d\mu\right)d\tau\right]\right\|d\theta ds$$

$$+\alpha\int_0^t\int_0^\infty \theta\,(t-s)^{\alpha-1}\zeta_\alpha(\theta)\left\|A^qQ((t-s)^\alpha\theta)\right\|\left\|\int_0^s h\left(s,\tau,u_0,\int_0^\tau k(\tau,\mu,u_0)d\mu\right)d\tau\right\|d\theta ds$$

Since $\int_0^\infty \theta^{1-q}\,\zeta_\alpha(\theta)d\theta\le 1$, so by using Lemma 1 (c), equations (10) and (11), above inequality gives

$$\|Fy(t)-A^qu_0\|\le\frac{\delta}{2}+M_q(1-q)^{-1}\{L\delta+B+L_1\delta a+L_1L_2\delta a^2+Ha\}a^{\alpha(1-q)}\le\delta.$$

Therefore, F maps S into itself. Moreover, if $y_1,y_2\in S$, then

$$\|Fy_1(t)-Fy_2(t)\|\le\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)\left\|A^qQ((t-s)^\alpha\theta)\right\|$$

$$\left\|f\big(s,A^{-q}y_1(s)\big)-f\big(s,A^{-q}y_2(s)\big)\right\|d\theta ds+\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)\left\|A^qQ((t-s)^\alpha\theta)\right\|$$

$$\left\|\left[\int_0^s h\left(s,\tau,A^{-q}y_1(\tau),\int_0^\tau k\big(\tau,\mu,A^{-q}y_1(\mu)\big)d\mu\right)\right.\right.$$

$$\left.\left.-\int_0^s h\left(s,\tau,A^{-q}y_2(\tau),\int_0^\tau k\big(\tau,\mu,A^{-q}y_2(\mu)\big)d\mu\right)d\tau\right]\right\|d\theta ds$$

$$\le M_qa^{\alpha(1-q)}(1-q)^{-1}L\|y_1-y_2\|_Y+M_qa^{\alpha(1-q)}(1-q)^{-1}L_1[(\|y_1-y_2\|_Y+L_2\|y_1-y_2\|_Ya)a]$$

$$\le M_qa^{\alpha(1-q)}(1-q)^{-1}[L+L_1(1+L_2a)a]\|y_1-y_2\|_Y$$

$$\le\frac{1}{2}\|y_1-y_2\|_Y,$$

which implies that

$$\|Fy_1-Fy_2\|_Y\le\frac{1}{2}\|y_1-y_2\|_Y.$$

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By the contraction mapping theorem, mapping F has a unique fixed point $y \in S$. This fixed point satisfies the integral equation

$$y(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) A^q u_0 d\theta + \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-s)^\alpha \theta)$$
$$\left[f\big(s, A^{-q} y(s)\big) + \int_0^s h\left(s, \tau, A^{-q} y(\tau), \int_0^\tau k\big(\tau, \mu, A^{-q} y(\mu)\big) d\mu \right) d\tau \right] d\theta ds. \tag{13}$$

From (7), (8) and the continuity of y it follows that

$$t \to f\big(t, A^{-q} y(t)\big)$$

and

$$t \to h\left(t, s, A^{-q} y(s), \int_0^s k\big(s, \tau, A^{-q} y(\tau)\big) d\tau \right)$$

are continuous on $[0, a]$, and, hence, there exist constants N_1 and N_2 such that

$$\left\| f\big(t, A^{-q} y(t)\big) \right\| \leq N_1 \tag{14}$$

and

$$\left\| h\left(t, s, A^{-q} y(s), \int_0^s k\big(s, \tau, A^{-q} y(\tau)\big) d\tau \right) \right\| \leq N_2. \tag{15}$$

By using the same method as in [15, Theorem 3.2], we can prove that $y(t)$ is locally H$\ddot{o}$lder continuous on $(0, a]$. Then there exist a constant C such that for every $t' > 0$, we have

$$\|y(t) - y(s)\| \leq C|t - s|^\gamma,$$

for all $0 \leq t' \leq t, s \leq a$. The local H$\ddot{o}$lder continuity of $t \to f\big(t, A^{-q} y(t)\big)$ follows from

$$\left\| f\big(t, A^{-q} y(t)\big) - f\big(s, A^{-q} y(s)\big) \right\| \leq L\big(|t-s|^\vartheta + \|y(t) - y(s)\|\big) \leq C_1\big(|t-s|^\vartheta + |t-s|^\gamma\big)$$

for some $C_1 > 0$ and the local H$\ddot{o}$lder continuity of

$$t \to h\left(t, s, A^{-q} y(s), \int_0^s k\big(s, \tau, A^{-q} y(\tau)\big) d\tau \right)$$

follows from

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$$\left\| h\left(t,s,A^{-q}y(s),\int_0^s k\big(s,\tau,A^{-q}y(\tau)\big)d\tau\right) - h\left(t,\mu,A^{-q}y(\mu),\int_0^s k\big(\mu,\phi,A^{-q}y(\phi)\big)d\phi\right)\right\|$$

$$\leq L_1\{|s-\mu|^{\vartheta} + \|y(s)-y(\mu)\| + L_2\big(|s-\mu|^{\vartheta} + |\tau-\phi|^{\vartheta} + \|y(\tau)-y(\phi)\|\big)a\}$$

$$\leq L_1\{|s-\mu|^{\vartheta} + |s-\mu|^{\gamma} + L_3\big(|s-\mu|^{\vartheta} + |\tau-\phi|^{\vartheta} + |\tau-\phi|^{\gamma}\big)a\}$$

for some $L_3 > 0$. Let y be a solution of (13). Consider the inhomogeneous initial value problem

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} + Au(t) = f\big(t,A^{-q}y(t)\big) + \int_0^t h\left(t,s,A^{-q}y(s),\int_0^s k\big(s,\tau,A^{-q}y(\tau)\big)d\tau\right)ds \tag{16}$$

$$u(0) = u_0. \tag{17}$$

This problem has a unique solution $u \in C^1\big((0,a]:X\big)$ [21], which is given by

$$u(t) = \int_0^{\infty} \zeta_{\alpha}(\theta)Q(t^{\alpha}\theta)u_0\,d\theta$$
$$+\alpha\int_0^t\int_0^{\infty} \theta(t-s)^{\alpha-1}\zeta_{\alpha}(\theta)Q((t-s)^{\alpha}\theta)\Bigg[f\big(s,A^{-q}y(s)\big)$$
$$+\int_0^s h\left(s,\tau,A^{-q}y(\tau),\int_0^{\tau} k\big(\tau,\mu,A^{-q}y(\mu)\big)d\mu\right)d\tau\Bigg]d\theta ds. \tag{18}$$

for $t > 0$, each term of (18) belongs to $D(A)$ and a *fortiori* in $D(A^q)$. Operating on both sides of (18) with A^q we find that

$$A^q u(t) = \int_0^{\infty} \zeta_{\alpha}(\theta)Q(t^{\alpha}\theta)A^q u_0\,d\theta$$
$$+\alpha\int_0^t\int_0^{\infty} \theta(t-s)^{\alpha-1}\zeta_{\alpha}(\theta)A^q Q((t-s)^{\alpha}\theta)\Bigg[f\big(s,A^{-q}y(s)\big)$$
$$+\int_0^s h\left(s,\tau,A^{-q}y(\tau),\int_0^{\tau} k\big(\tau,\mu,A^{-q}y(\mu)\big)d\mu\right)d\tau\Bigg]d\theta ds. \tag{19}$$

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From (13) the right hand side of (19) equals $\boldsymbol{y}(\boldsymbol{t})$ and therefore $\boldsymbol{u}(\boldsymbol{t}) = \boldsymbol{A}^{-q}\boldsymbol{y}(\boldsymbol{t})$ and by (18), $\boldsymbol{u}$ is a $\boldsymbol{C}^{\mathbf{1}}\big((\mathbf{0}, \boldsymbol{a}]: \boldsymbol{X}\big)$ solution of $(\mathbf{1}) - (\mathbf{2})$. The uniqueness of $\boldsymbol{u}$ follows from the uniqueness of the solutions of (13) and $(\mathbf{16}) - (\mathbf{17})$. Hence, the theorem is proved.

References

[1] Balachandran, K. and Chandrasekaran, M., Existence of solutions of nonlinear integrodifferential equation with nonlocal condition, *Journal of Applied Mathematics and Stochastic Analysis*, 10(3) (**1997**): 279-288.

[2] Bahuguna, D., Integrodifferential equations with analytic semigroups, *J. App. Math. and Stochastic Analysis*, 16(2)(**2003**): 177-189.

[3] Kolodziej, Katarzyna, Existence and uniqueness of solutions of a semilinear functional – differential evolution nonlocal Cauchy problem, *Journal of Applied Mathematics and Stochastic Analysis*, 13(2)(**2000**): 171-179.

[4] Kumar, Kamalendra and Kumar, Rakesh, Nonlocal Cauchy problem for integrodifferential equations of Sobolev type in Banach Space, *Matematika*, 29(2)(**2013**): 179-186.

[5] Kumar, Kamalendra and Kumar, Rakesh, Existence of solutions of nonlinear delayed integrodifferential equation with nonlocal condition, in the proceedings "*Advance in Applied Mathematics and Computational Physics*" (Eds. Rakesh Kumar et.al.), World Education Publishers, New Delhi, India (**2012**): 227-235.

[6] Lakshmikantham, V. , Leela, S. and Devi, J. V., *Theory of fractional dynamic systems*, Cambridge Scientific Publication, Cambridge, UK, (**2009**).

[7] Miller, K. S. and Ross, B., *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, (**1993**).

[8] Podlubny, I., *Fractional differenatial equations*, Academic Press, New York, (**1999**).

[9] Ahmed, H. M., Semilinear differential equations of fractional order with nonlocal conditions, *Int. J. of Appl. Math. And Mech.* 5(3)(**2009**): 69-74.

[10] Li, F., Mild solution for fractional differential equations with nonlocal conditions, *Adv. Difference Equ.*, Article ID 287861, 9 pages, (**2010)**.

[11] Mophou, G. M., N' Guerekata, G. M., Mild solution for semilinear fractional differential equations, *Electron. J. Differ. Equ.* 21(**2009**): 1-9.

[12] Matar, M. M., Existence and uniqueness of solutions to fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions, *EJDE*, 2009(155)(**2009**): 1-7.

[13] Gelfand, I.M., Shilov, G.E., *In: Generalized functions*, Vol. 1, Moscow: Nauka; **1959**.

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[14] Debbouche, A. and El-Borai, M.M., Weak almost periodic and optimal mild solutions of fractional evolution equations, *Electronic J. Diff. Eqns.*, 46(**2009**): 1-8.

[15] El-Borai, M.M., Some probability densities and fundamental solutions of fractional evolution equations, *Chaos, Solitons and Fractals*, 14(3)(**2002**): 433-440.

[16] El-Borai, M.M., The fundamental solutions for fractional evolution equations of parabolic type, *J. App. Math. and Stochastic Analysis*,2004(3)(**2004**): 197-211.

[17] El-Borai, M.M., Semigroups and some nonlinear fractional differential equations, *App. Math. Comput.*, 149(3)(**2004**): 823-831.

[18] El-Borai, M.M., On some fractional evolution equations with nonlocal conditions, *Int. J. Pure Appl. Math.*, 24(3)(**2005**): 405-413.

[19] El-Borai, M.M., On some stochastic fractional integro-differential equations, *Advance in Dyn. Sys. Appl.* ISSN 0973-5321, 1(1)(**2006**): 49-57.

[20] El-Borai, M.M. and Debbouche, A., On some fractional integrodifferential equations with analytic semigroups, Int. J. Contemp. Math. Sciences, 4(28)(**2009**): 1361-1371.

[21] Pazy, A., *Semigroup of Linear Operators and Application to Partial Differential Equations*, Springer-Verlag, New York, (**1983)**.

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