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Article

A New Three Parameter Consul Kumaraswamy Distribution with Application

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Abstract: In the present paper we construct a new three parameter distribution which is obtained by compounding a Consul distribution with Kumaraswamy distribution. The new distribution so obtained is known as Consul Kumaraswamy distribution (CKSD) which can be nested to different compound distributions. Furthermore, some mathematical properties such as factorial moments, mean, variance and coefficient of variation of some compound distributions have also been discussed. The estimation of parameters of the proposed distribution has been obtained via maximum likelihood estimation method. Finally the potentiality of proposed distribution is justified by using it to model the real life data set.

Keywords: Consul distribution, Kumaraswamy distribution, compound distribution, factorial moment

Mathematics Subject Classification: 60E05

1. Introduction

From the last few decades researchers are busy to obtain new probability distributions by using different techniques such as compounding, T-X family, transmutation etc. but compounding of probability distribution has received maximum attention which is an innovative and sound technique to

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obtain new probability distributions. The compounding of probability distributions enables us to obtain both discrete as well as continuous distribution.

Compound distribution arises when all or some parameters of a distribution known as parent distribution vary according to some probability distribution called the compounding distribution for instance negative binomial distribution can be obtained from Poisson distribution when its parameter λ follows gamma distribution. If the parent distribution is discrete then resultant compound distribution will also be discrete and if the parent distribution is continuous then resultant compound distribution will also be continuous i,e. the support of the original (parent) distribution determines the support of compound distributions.

In several research papers it has been found that compound distributions are very flexible and can be used efficiently to model different types of data sets. With this in mind many compound probability distributions have been constructed. Sankaran [10] obtained a compound of Poisson distribution with that of Lindley distribution, Zamani and Ismail constructed a new compound distribution by compounding negative binomial with one parameter Lindley distribution that provides good fit for count data where the probability at zero has a large value [11]. Researchers like Adil and Jan obtained several compound distributions for instance, a compound of zero truncated generalized negative binomial distribution with generalized beta distribution [3], a compound of Geeta distribution with generalized beta distribution [4] and compound of Consul distribution with generalized beta distribution [5] recently Adil and Jan explored a mixture of generalized negative binomial distribution with that of generalized exponential distribution which contains several compound distributions as its sub cases and proved that this particular model is better in comparison to others when it comes to fit observed count data set [1]. Most recently Adil and Jan constructed a new lifetime distribution and some count data models with wide applications in real life [2, 6,7].

2. Methods

2.1. *Consul Distribution (CD)*

Consul distribution introduced by Consul and Shenton was modified by Islam and Consul (1990) who derived it as a bunching model in traffic flow through the branching process and also discussed its applications to automobile insurance claims and vehicle bunch size data [8].

Suppose a queue is initiated with one member and has traffic intensity with binomial arrivals, given by generating function $g(t) = (1 - p + pt)^m$ and constant service time. Then the probability that exactly x members will be served before the queue vanishes is given by Consul distribution with probability mass function given

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$$f_1(x;m,p)=\begin{cases}\frac{1}{x}\binom{mx}{x-1}p^{x-1}q^{mx-x+1} & ;\ x=1,2,\ldots\\ 0 & ;\ otherwise\end{cases} \tag{1}$$

where $0<p<1$ and $1\le m\le\frac{1}{p}$.

Consul distribution reduces to the geometric distribution when $m=1$ in (1).

2.2. Kumaraswamy Distribution (KSD)

Kumaraswamy distribution is a two parameter continuous probability distribution that has obtained by Kumaraswamy but unfortunately this distribution is not very popular among statisticians because researchers have not analyzed and investigated it systematically in much detail. Kumaraswmy distribution is similar to the beta distribution but unlike beta distribution it has a closed form of cumulative distribution function which makes it very simple to deal with. For more detailed properties one can see references [9] and Jones [12].

A random variable X is said to have a Kumaraswamy distribution (KSD) if its pdf is given by

$$f_2(X;\alpha,\beta)=\alpha\beta\ x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1},\ 0<x<1 \tag{2}$$

where $\alpha,\beta>0$ are shape parameters.

In the Consul distribution the parameters m and p are fixed but here we have considered a problem in which the parameter m is fixed but the probability parameter p is itself a random variable following Kumarasawmy distribution (2), in that case the probability that exactly x members will be served before the queue vanishes is given by the compound of Consul distribution with Kumarasawmy distribution. Here is the definition that will expose the probability function of the proposed distribution.

2.3. Definition of Proposed Distribution

If a random variable X follows Consul distribution with parameters m and p where the parameter m is fixed but p instead of being a fixed constant is also a random variable following Kumarasawamy distribution then determining the distribution that results from marginalizing over p will be known as a compound of Consul distribution with that of Kumarasawmy distribution.

***Theorem 2.3.1*:** The probability function of a compound of CD (m,p) with KSD (α,β) is given by

$$f_{CKSD}(X;m,\alpha,\beta)=\binom{mx}{x-1}\sum_{j=0}^{\infty}\binom{mx-x+1}{j}(-1)^j\,B\left(\beta,\frac{x+j-1}{\alpha}+1\right)$$

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where $x = 1, 2, \ldots, m, \alpha, \beta > 0$.

Proof: With the help of definition of proposed distribution the probability function of a compound of CD (m, p) with KSD (α, β) can be obtained as

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \int_0^1 f_1\left(x \mid p\right) f_2\left(p\right) dp$$

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\alpha\beta}{x}\binom{mx}{x-1} \int_0^1 p^{x+\alpha-2}\left(1-p\right)^{mx-x+1}\left(1-p^{\alpha}\right)^{\beta-1} dp \tag{3}$$

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\alpha\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{\infty}\binom{mx-x+1}{j}(-1)^j \int_0^1 p^{x+j+\alpha-2}\left(1-p^{\alpha}\right)^{\beta-1} dp$$

Substituting, $1 - p^{\alpha} = z$ we get

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{\infty}\binom{mx-x+1}{j}(-1)^j \int_0^1 z^{\beta-1}\left(1-z\right)^{\frac{x+j+\alpha-1}{\alpha}} dz$$

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{\infty}\binom{mx-x+1}{j}(-1)^j B\left(\beta, \frac{x+j+\alpha-1}{\alpha}+1\right) \tag{4}$$

$B(\cdot)$ refers to the beta function defined by $B(r, s) = \Gamma(r)\Gamma(s)/\Gamma(r+s)$, $r, s > 0$ and $x = 1, 2, \ldots, m, \alpha, \beta > 0$. From here a random X variable following a compound of CD with KSD will be symbolized by CKSD (m, α, β). It may be noted here that equation (4) is a valid pmf since it has been obtained by using a well-known stochastic compound formula which gives rise to unconditional pmf.

In the special case if $m \in N$ the above probability function takes the simpler rearranged form as

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{mx-x+1}\binom{mx-x+1}{j}(-1)^j B\left(\beta, \frac{x+j+\alpha-1}{\alpha}+1\right) \tag{5}$$

where $x = 1, 2, \ldots, \alpha, \beta > 0$ and $m \in N$. Alternatively we can also proceed from (3) as

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\alpha\beta}{x}\binom{mx}{x-1} \int_0^1 p^{x+\alpha-2}\left(1-p\right)^{mx-x+1}\left(1-p^{\alpha}\right)^{\beta-1} dp$$

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\alpha\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^j \int_0^1 p^{x+\alpha j+\alpha-2}\left(1-p\right)^{mx-x+1} dp$$

$$f_{CKSD}\left(X; m, \alpha, \beta\right) = \frac{\alpha\beta}{x}\binom{mx}{x-1}\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^j B\left(x+\alpha+\alpha j-1, mx-x+2\right)$$

where $x = 1, 2, \ldots, m, \alpha, \beta > 0$. This gives another form of pmf of CKSD (m, α, β).

2.4. Some Nested Distributions of CKSD

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In this section it will be shown that CKSD can be nested to different compound probability distributions for specific parameter setting

Proposition 2.4.1: If X ~ $\text{CKSD}(m,\alpha,\beta)$, then by setting $m=1$ we obtain compound of geometric distribution with Kumaraswamy distribution.

Proof: Since for $m=1$ in CD we obtain geometric distribution. Hence a compound of geometric distribution with Kumaraswmy distribution is followed from (5) by simply substituting $m=1$ in it. Therefore we have

$$f_{GKSD}(X;\alpha,\beta)=\frac{\beta}{x}\binom{x}{x-1}\sum_{j=0}^{1}\binom{1}{j}(-1)^j B\left(\beta,\frac{x+j+\alpha-1}{\alpha}+1\right)$$

$$f_{GKSD}(X;\alpha,\beta)=\beta\sum_{j=0}^{1}\binom{1}{j}(-1)^j B\left(\beta,\frac{x+j+\alpha-1}{\alpha}+1\right)$$

Where $x=1,2,...,\alpha,\beta>0$

***Proposition 2.4.2*:** If X ~ $\text{CKSD}(m,\alpha,\beta)$, then by setting $\alpha=\beta=1$ we obtain compound of CD distribution with uniform distribution.

Proof: For $\alpha=\beta=1$, Kumaraswamy distribution reduces to uniform distribution. Hence a compound of CD with uniform distribution is followed from (5) when we substitute $\alpha=\beta=1$ in it.

$$f_{CUD}(X;m)=\frac{1}{x}\binom{mx}{x-1}\sum_{j=0}^{mx-x+1}\binom{mx-x+1}{j}(-1)^j B(1,x+j+1)$$

Where $x=1,2,...,m>0$

***Proposition 2.4.3*:** If X ~ $\text{CKSD}(m,\alpha,\beta)$, then by setting $m=\alpha=\beta=1$ we obtain compound of geometric distribution with uniform distribution with parameters

Proof: The proof of the above proposition is followed from (5) by substituting $m=\alpha=\beta=1$ in it

$$f_{GUD}(X)=\frac{1}{x}\binom{x}{x-1}\sum_{j=0}^{1}\binom{1}{j}(-1)^j B(1,x+j+1)$$

$$f_{GUD}(X)=\sum_{j=0}^{1}\binom{1}{j}(-1)^j B(1,x+j+1), \qquad x=1,2,...$$

2.5. Factorial Moments

We hardly emphasize on the necessity and importance of factorial moments in any statistical analysis especially in applied work. Some of the important features and characteristics of a distribution can be studied through factorial moments (e.g mean, variance, standard deviation etc).In this section we obtain factorial moments of some compound distributions through which some of the important mathematical properties will also be discussed.

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If $X \mid p \sim \mathrm{GD}(X;p)$, where p follows $\mathrm{KSD}(P;\alpha,\beta)$ then

$$\mu_{[l]}(X) = E_{\lambda}\left[m_l\left(X \mid p\right)\right] \tag{6}$$

is called a factorial moment of order l of a compound of GD with KSD where $m_l\left(X \mid p\right)$ is the 1st order factorial moment of GD.

***Theorem 2.5.1*:** The factorial moment of order l of a compound of GD with KSD distribution is given by the expression

$$\mu_{[l]}(X) = l!\beta \sum_{j=0}^{l} \binom{l}{j} (-1)^j B\left(\beta, \frac{j-l}{\alpha}+1\right)$$

Where, $x=1,2,...,\alpha,\beta>0$

Proof: First order factorial of GD is

$$m_{[l]}(X,P) = l!\left(\frac{q}{p}\right)^l$$

Therefore factorial moment of a compound of GD with KSD by using the definition (6) can be obtained as

$$\mu_{[l]}(X) = \int_0^1 l!\left(\frac{q}{p}\right)^l \alpha\beta\, p^{\alpha-1}\left(1-p^{\alpha}\right)^{\beta-1} dp$$

$$\mu_{[l]}(X) = l!\alpha\beta \int_0^1 (1-p)^l\, p^{\alpha-l-1}\left(1-p^{\alpha}\right)^{\beta-1} dp$$

$$\mu_{[l]}(X) = l!\alpha\beta \sum_{j=0}^{l} \binom{l}{j} (-1)^j \int_0^1 p^{\alpha+j-l-1}\left(1-p^{\alpha}\right)^{\beta-1} dp$$

Substituting, $1-p^{\alpha}=z$, we get

$$\mu_{[l]}(X) = l!\alpha\beta \sum_{j=0}^{l} \binom{l}{j} (-1)^j \int_0^1 z^{\beta-1}\left(1-z\right)^{\frac{j-l}{\alpha}} dz$$

$$\mu_{[l]}(X) = l!\beta \sum_{j=0}^{l} \binom{l}{j} (-1)^j B\left(\beta, \frac{j-l}{\alpha}+1\right) \tag{7}$$

where, $x=1,2,...,\alpha,\beta>0$.

For $l=1$ in (7) we obtain mean of a compound of GD with KSD

$$\mu_{[1]}(X) = \beta \sum_{j=0}^{1} \binom{1}{j} (-1)^j B\left(\beta, \frac{j-1}{\alpha}+1\right)$$

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$$\mu_{[1]}(X)=\sum_{j=0}^{1}\binom{1}{j}(-1)^{j}\,\frac{\Gamma(\beta+1)\Gamma\left(\frac{j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{j-1}{\alpha}+1\right)}$$

$$\text{Mean }=\mu_{[1]}(X)=\frac{\beta\Gamma(\beta)\Gamma\left(\frac{\alpha-1}{\alpha}\right)}{\left(\beta+\frac{1}{\alpha}\right)\Gamma\left(\beta+\frac{1}{\alpha}\right)}-1=\upsilon_{1}$$

$$\mu_{[2]}(X)=2\sum_{j=0}^{2}\binom{2}{j}(-1)^{j}\,\frac{\Gamma(\beta+1)\Gamma\left(\frac{j-2}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{j-2}{\alpha}+1\right)}$$

$$\mu_{[2]}(X)=2\left(\frac{\beta\Gamma(\beta)\Gamma\left(\frac{\alpha-2}{\alpha}\right)}{\Gamma\left(\beta-\frac{2}{\alpha}+1\right)}-2\frac{\beta\Gamma(\beta)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(\beta-\frac{1}{\alpha}+1\right)}+1\right)=2\upsilon_{2}$$

$$E(X^{2})=\mu_{[2]}(X)+\mu_{[1]}(X)=2\upsilon_{2}+\upsilon_{1}$$

$$V(X)=2\upsilon_{2}+\upsilon_{1}-\upsilon_{1}^{2}$$

$SD=\sqrt{2\upsilon_{2}+\upsilon_{1}-\upsilon_{1}^{2}}$ and coefficient of variation $CV=\frac{\sqrt{2\upsilon_{2}+\upsilon_{1}-\upsilon_{1}^{2}}}{\upsilon_{1}}$

Corollary 2.5.2: The factorial moment of order l of a compound of GD with uniform distribution is given by the expression

$$\mu_{[l]}(X)=l!\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}B(1,\,j-l+1),\qquad x=1,2,3,\ldots$$

<u>Proof:</u> For $\alpha=\beta=1$, Kumaraswamy distribution reduces to uniform distribution. Therefore the proof of the above corollary is followed from (7) when we substitute $\alpha=\beta=1$in it.

$$\mu_{[l]}(X)=l!\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}B(1,\,j-l+1),\quad x=1,2,\ldots$$

2.6. Parameter Estimation

In this particular section the estimation of parameters of the proposed distribution will be discussed in detail via maximum likelihood estimation method. Let $(x_1,x_2,\ldots x_N)^T$ be a random sample from CKSD (m,α,β) with unknown parameter vector $\Theta=(m,\alpha,\beta)^T$. In order to find MLE of the proposed distribution we consider its rearranged pmf (5) because of its simple structure as the pmf

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defined in (4) involves an infinite series that results in too many iterations and therefore unnecessary computer time. Hence the log-likelihood function of (5) is given by

$$£(x,m,\alpha,\beta)=\log L(x,m,\alpha,\beta)=n\log\beta+\sum_{i=1}^{n}\log\left(\Gamma(mx_i+1)\right)-\sum_{i=1}^{n}\log\left(\Gamma(mx_i-x_i+2)\right)-\sum_{i=1}^{n}\log\left(\Gamma(x_i+1)\right)+$$
$$\sum_{i=1}^{n}\log\left(\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^{j}\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}\right) \tag{8}$$

The first order conditions for finding the optimal values of the parameters obtained by differentiating (8) with respect m,α *and* β respectively gives rise to the following differential equations

$$\frac{\partial £}{\partial\alpha}=\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^{j}\;\Gamma(\beta)\frac{\partial}{\partial\alpha}\left(\frac{\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}\right)}{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^{j}\;\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}}\right)$$

$$\frac{\partial £}{\partial m}=\sum_{i=1}^{n}\frac{\frac{\partial}{\partial m}\left(\Gamma(mx_i+1)\right)}{\Gamma(mx_i+1)}-\sum_{i=1}^{n}\frac{\frac{\partial}{\partial m}\left(\Gamma(mx_i-x_i+2)\right)}{\Gamma(mx_i-x_i+2)}+$$
$$\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial m}\left(\sum_{j=0}^{mx_i-x_i+1}\frac{\Gamma(mx_i-x_i+2)}{\Gamma(j+1)\Gamma(mx_i-x_i-j+2)}(-1)^{j}\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}\right)}{\sum_{j=0}^{mx_i-x_i+1_i}\binom{mx_i-x_i+1}{j}(-1)^{j}\;\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}}\right)$$

$$\frac{\partial £}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^{j}\;\frac{\partial}{\partial\beta}\left(\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}\right)}{\sum_{j=0}^{mx_i-x_i+1}\binom{mx_i-x_i+1}{j}(-1)^{j}\;\frac{\Gamma(\beta)\Gamma\left(\frac{x_i+j-1}{\alpha}+1\right)}{\Gamma\left(\beta+\frac{x_i+j-1}{\alpha}+1\right)}}\right)$$

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The above equations are very difficult to solve for analytical solution, therefore $\hat{m}, \hat{\alpha}$ and $\hat{\beta}$ will be obtained by maximizing the log likelihood function numerically using Newton-Raphson method in R software which is a very powerful technique for solving equations iteratively and numerically.

3. Results and Discussion

In this section we will explore the applicability of the proposed Consul Kumaraswamy distribution by using a real data set on bunching traffic in Australian rural highways which have been taken from Taylor et al [13].The data which appears in the first two columns of table 1 gives bunch size with observed corresponding frequency and the data which appears in the 3rd and 4th column of this table is the fitted CD and proposed CKSD.

Table 1: Bunch size frequency distribution of Australian rural highways (Taylor et al., 1974)

Number of mites per leaf	**Observed Frequency**	**Fitted Distribution**	
		CD	CKSD
1	127	125.42	125.64
2	53	58.83	59.27
3	29	29.07	29.60
4	21	14.84	14.51
5	5	9.29	6.77
6	4	4.12	2.93
7	1	2.22	1.14
8	5	1.21	5.14
Total	245	245	245
Parameter Estimation		$\hat{m} = 1.12$ $\hat{p} = 0.45$	$\hat{m} = 0.80$ $\hat{\alpha} = 0.68,\ \hat{\beta} = 0.95$
Chi-Square Estimate		5.93	4.12
DF		3	2
p-value		0.116	0.127

The proposed distributions fitting graphs of models in table 1 are presented in figs 1 and 2.

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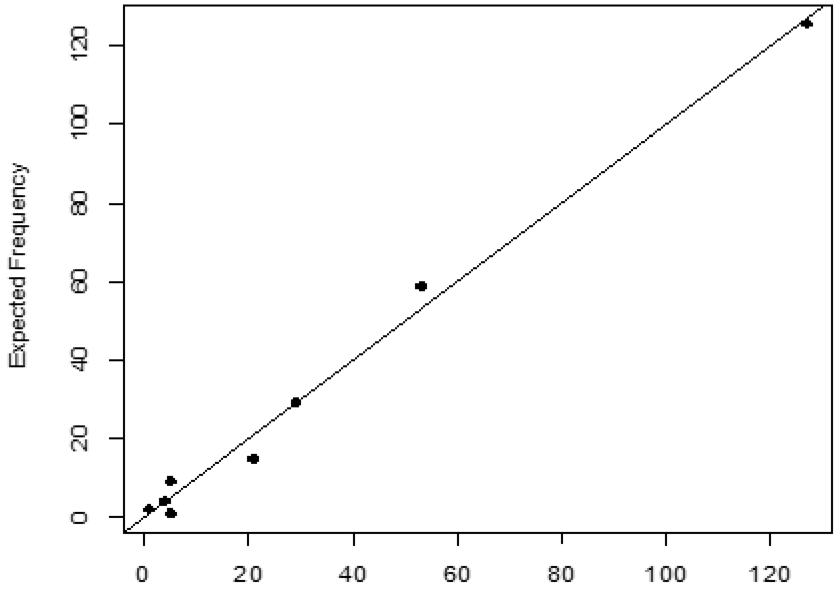

Fig. 1: Observed frequency fitting of CD model in table 1

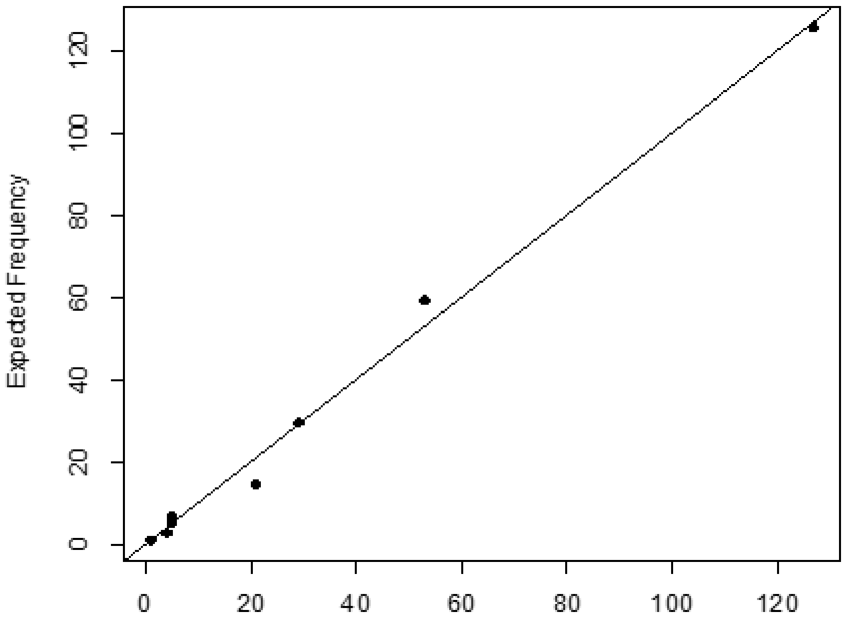

Fig. 2: Observed frequency fitting of CKSD model in table 1

4. Conclusion

In this paper we have proposed a new three parameter Consul Kumaraswamy distribution by using a compounding mechanism. The probability mass function of proposed CKSD has two different parameter arrangements one of infinite series and other with a finite series. We have also shown that proposed distribution embodies some new compound distributions. Factorial moments of some compound distribution have also been discussed. Parameters of the proposed distribution have been

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obtained by means of maximum likelihood estimation technique. In the end the potentiality of the proposed distribution have been tested by fitting it to the real data set and it is quite clear and evident from the results that proposed model provides a better fit in comparison to Consul distribution.

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