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Article

Explicit Solutions of a Generalized Hirota-Satsuma Equation Using Darboux Transformation

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Abstract: A modified version of the generalized Hirota-Satsuma equation is solved analytically using Darboux transformations (DT). We start with the Lax pair of this equation and apply DT. This leads to another solvable pair containing a new eigenfunction that is a solution of the equation. Several seeds solutions are tested as well as one and two solitons forms are obtained using DT. A suitable choice of the seed fields leads to new solutions.

Keywords: Solitons; Hirota-Satsuma equation; Darboux Transformation; exact solutions; Singular Manifold method; Lax pair.

1. Introduction

The wave observed in plasma, elastic media, optical fibers, fluid dynamics are described by nonlinear partial differential equations. In the past decades, several methods for obtaining analytic solutions of nonlinear partial differential equations (NPDEs) have been presented, such as the inverse scattering method[1], Hirota's method [2,3], the Backlund transformation [4,5] and Darboux transformation [6-10], Painlevé expansions [11], homogenous balance method [12,13], Jacobi elliptic function [14, 15], extended tanh-function methods[16-18], extended F-expansion methods [19-20], Adomain methods [21], Exp -function methods [22] and finally the Mapping method [23-24] .

In this paper, we solve using Darboux Transformation method the generalized Hirota–Satsuma equation in three dimensions (3D) which described as follows:

$$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = h_{yz} \tag{1}$$

This equation describes the flow of an incompressible fluid. Using the Singular Manifold Method (SMM), Estevez *et al* [31] derive its Lax pair in the form of:

$$-\psi_y + \psi_{xxx} + 3h_x\psi_x + \frac{3}{2}h_{xx}\psi = 0 \quad (2)$$

$$2h_z\psi_{xz} - h_{xz}\psi_z + 2h_z^2\psi = 0 \quad (3)$$

where h(x,y,t) is the wave amplitude and $\psi(x,y,z)$ in the system (2) and (3) is eigenfunction. The present work is organized as follows; section 2, is devoted to the mathematical formulation of the problem .We start with an initial solution h and recursively obtain via the system eigenfunctions ψ, ψ_1 , an improved solution h[1]. Applying DT, N-times, produces N-soliton solutions. In Section 3, we explicitly detail, the explicit solitary wave solutions for different seeds form and plot them. In Section 4, two soliton solutions are derived and plotted, applying the DT method. The paper ends with a conclusion, in section 5.

2. Mathematical Formulation

Darboux transformation is a recursive algorithm; deriving a series of explicit solutions from a trivial one. Applying it to the Lax pair (2) and (3) results in two eigenfunction ψ, ψ_1. These are used together, with an initial seed solution h in the following equations;

$$\psi[1] = \left(\frac{d}{dx} - \frac{\psi'_1}{\psi_1}\right)\psi \quad (4)$$

$$h[1] = h[0] + \frac{\psi'_1}{\psi_1} \quad (5)$$

where $\psi[1]$ satisfies (2) and (3) and $h[1]$ is a new solution (one soliton) for equation (1). Replacing for $\psi[1]$ in (2) and (3) yields:

$$-\psi_y[1] + \psi_{xxx}[1] + 3h_x[1]\,\psi_x[1] + \frac{3}{2}h_{xx}[1]\,\psi[1] = 0 \quad (6)$$

$$2h_z[1]\,\psi_{xz}[1] - h_{xz}[1]\,\psi_z[1] + 2h_z^2[1]\,\psi[1] = 0 \quad (7)$$

Applying DT, once more to get the two solitons solution we have:

$$\psi[2] = \left[\frac{d}{dx} - \frac{\psi'_2[1]}{\psi_2[1]}\right]\psi[1] \quad (8)$$

$$h[2] = h[1] + \frac{d}{dx}\ln\psi_2[1] \quad (9)$$

where $\psi_2[1]$ is defined as:

$$\psi_2[1] = \left[\frac{d}{dx} - \frac{\psi'_1}{\psi_1}\right]\psi_2 \quad (10)$$

where ψ_2 is additional solution of (2) and (3) using a different constant of integration. From (8) into equations (6) and (7) we obtain:

$$-\psi_y[2] + \psi_{xxx}[2] + 3h_x[2]\,\psi_x[2] + \frac{3}{2}h_{xx}[2]\,\psi[2] = 0 \tag{11}$$

$$2h_z[2]\psi_{xz}[2] - h_{xz}[2]\psi_z[2] + 2h_z^2[2]\,\psi[2] = 0 \tag{12}$$

where $h[2]$ is a new solution (two solitons) of equation (1). Applying DT N-time gives the following forms for $\psi[N], h[N]$:

$$\psi[N] = \frac{W(\psi_1,\psi_2,\dots,\psi_N,\psi)}{W(\psi_1,\psi_2,\dots,\psi_N)} \tag{13}$$

$$h[N] = h[0] + \frac{d}{dx}\ln W(\psi_1,\psi_2,\dots,\psi_N) \tag{14}$$

where *W* is the Wronskian of the eigenfunctions; $\psi_1,\psi_2,\dots,\psi_N,\psi$ **.**

3. Solitary Wave Solution (one soliton)

This section gives a single soliton (solitary wave) solution for both the nonlinear equation (1) and its Lax pair (2) and (3). To simplify the solution of this system, we use a simple seed field (h). Some seed fields are chosen and the explicit solutions are given below.

3.1. First Initial (seed) Solution

Consider an initial wave form:

$$h[0] = x + y + z \tag{15}$$

Substituting *h[0]* into equations (2) and (3) gives:

$$-\psi_y + \psi_{xxx} + 3\psi_x = 0 \tag{16}$$

$$\psi_{xz} + \psi = 0 \tag{17}$$

Let in equation (16) the solitary wave solution be; $\psi(x,y,z) = \phi(\zeta)$, where $\zeta = x + y + \frac{z}{2}$

Thus equation (16) reduces to

$$\phi_{\zeta\zeta\zeta} + 2\phi_\zeta = 0, \phi = \phi(\zeta) \tag{18}$$

Integrating with respect to ζ , we obtain:

$$\phi_{\zeta\zeta} + 2\,\phi = c \tag{19}$$

where, c is an arbitrary constant of integration. As the boundary conditions for solitary wave are; $\phi, \phi_\zeta, \phi_{\zeta\zeta} \to 0$ as $\zeta \to \pm\infty$, thus c =0. Hence the solution of equation (19) will be;

$$\phi(\zeta) = k_1 e^{\sqrt{2}\,i\zeta} + k_2 e^{-\sqrt{2}\,i\zeta} \tag{20}$$

So,

$$\phi(x,y,z) = \psi(x,y,z) = k_1 e^{i\sqrt{2}(x+y+\frac{1}{2}z)} + k_2 e^{-i\sqrt{2}(x+y+\frac{1}{2}z)} \tag{21}$$

The two solutions $\psi,\ \psi_1$ of eq. (4) are obtained by choosing $k_1 = \frac{1}{2}$, $k_2 = \mp\frac{1}{2}$.

For $k_1 = \frac{1}{2}$, $k_2 = \frac{1}{2}$ we obtain:

$$\psi(x, y, z) = cos\left(\sqrt{2}(x + y + \tfrac{1}{2}z)\right) \tag{22}$$

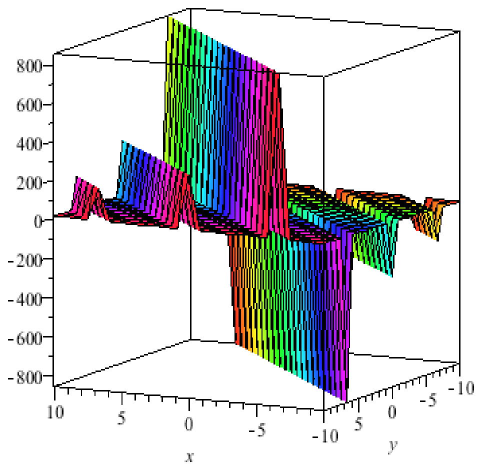
For $k_1 = \frac{1}{2}$, $k_2 = \frac{-1}{2}$ we have:

$$\psi_1(x, y, z) = sin\left(\sqrt{2}(x + y + \tfrac{1}{2}z)\right) \tag{23}$$

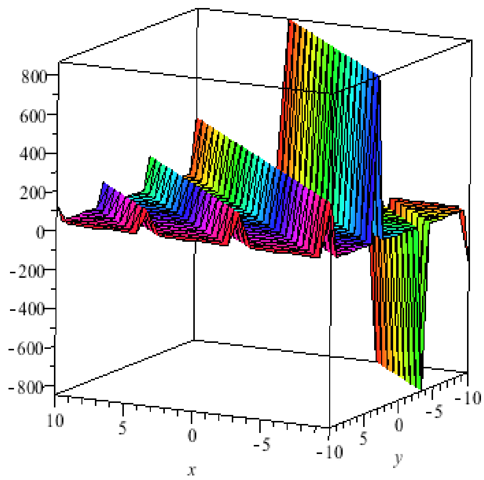
and $h[1]$ in (5) reduces to:

$$h[1] = x + y + z - \sqrt{2}\, tan\left(\sqrt{2}(x + y + \tfrac{1}{2}z)\right) \tag{24}$$

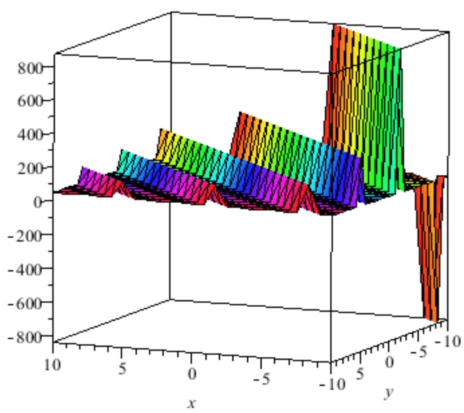
h[1] which is the solitary solution of equation (1). Figure 1 show the solitary wave solution for the non-linear equation (1) for $z = 5, 20, 30, 40$



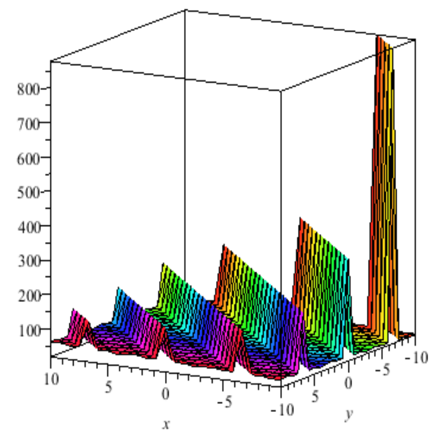
a. h[1] for $h[0] = x + y + z$, z=5



b. h[1] for $h[0] = x + y + z$, $z = 20$



c. h[1] for $h[0] = x + y + z$, z=30



d. h[1] for $h[0] = x + y + z$, z=40

Fig. 1. Solitary wave solution for equation (1) for different z's

4. Two-Soliton Solution

To derive a two-soliton solution; *h[2]*, we apply the DT formula for two solitons solution, formula (8) and (9) that can be written as ;

$$\psi[2] = \frac{W(\psi_1,\psi_2,\psi)}{W(\psi_1,\psi_2)} \tag{25}$$

$$h[2] = h[0] + \frac{d}{dx}\ln W(\psi_1,\psi_2) \ = h[0] + \sigma + \sigma_1 \tag{26}$$

where W is the wronskian of three eigenfunctions; ψ, ψ_1, ψ_2 , while *h[0]* is the seed solution. This is the two-soliton solution. This solution necessitates three eigen-values; (ψ, ψ_1, ψ_2), we assume for ψ_2 a form similar to ψ, ψ_1

$$\psi_2(x,y,z) = sin(x + 2y + z) \tag{27}$$

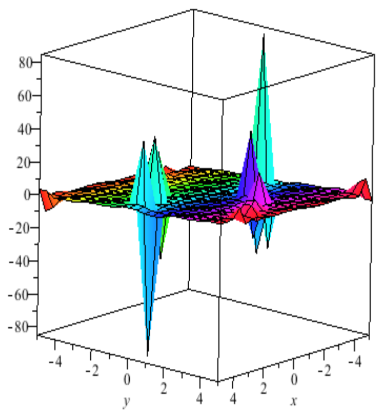
We then solve the problem for two different seeds values.

4.1. First Seed Solution

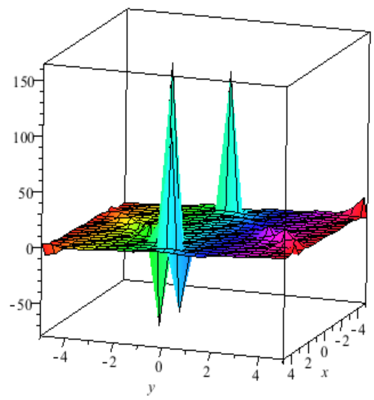
Replacing for $h[0] = x + y + z$ in (26) gives two-soliton solution for the nonlinear equation (1) with the explicit form;

$$h[2] = x + y + z - \sqrt{2}\, tan\left(\sqrt{2}\left(x + y + \frac{1}{2}z\right)\right) + \frac{-\cos(\sqrt{2}\left(x+y+\frac{1}{2}z\right))^2 \sin(x+2y+z)+2sin(x+2y+z)+\sqrt{2}\sin(2\sqrt{2}\left(x+y+\frac{1}{2}z\right))cos(x+2y+z)}{\cos(\sqrt{2}\left(x+y+\frac{1}{2}z\right))^2 cos(x+2y+z)+\frac{1}{\sqrt{2}}\sin(2\sqrt{2}\left(x+y+\frac{1}{2}z\right))sin(x+2y+z)} \tag{28}$$

This solution is plotted in Fig.2 for initial solution $h[0] = x + y + z$



a. h[2] for $h[0] = x + y + z$, z=0



b. h[2] for $h[0] = x + y + z$, z=0.1

Fig. 2. Two solitons solution for equation (1)

4.2. Second Seed Solution

Consider initial field of the form

$$h[0] = xy + \frac{1}{z} \tag{29}$$

Substituting *h[0]* from (29) in system (2, 3) reduces it to the form;

$$\psi_{xz} - \frac{1}{z^2}\psi = 0 \tag{30}$$

$$\psi_{xxx} - \psi_y + 3y\psi_x = 0 \tag{31}$$

Solving eq. (30), eq. (31) together gives;

$$\psi_1(x,y,z) = 0.5e^{(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z})} + 0.5e^{-(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z})} \tag{32}$$

We can choose in the same $\psi_2(x,y,z)$ form

$$\psi_2(x,y,z) = 0.5e^{(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z})} + 0.5e^{-(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z})} \tag{33}$$

Using eq. (32), we get;

$$\sigma = \frac{\psi_1{}'(x,y,z)}{\psi_1(x,y,z)} = \lambda_1 \tanh(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}) \tag{34}$$

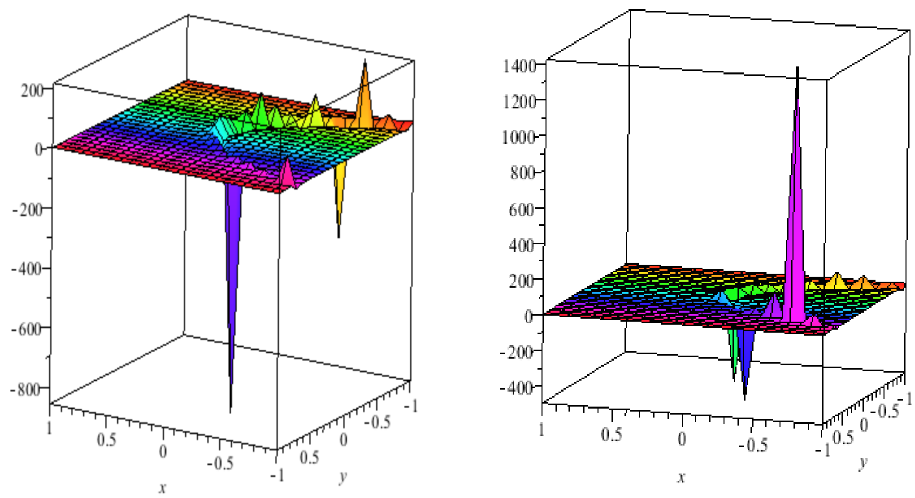
Using eq. (33), we get;

$$\sigma_1 = \frac{\psi_2'[1]}{\psi_2[1]} = \frac{\lambda_2^2 - \lambda_1^2 + \lambda_2{}^2\, tanh^2\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right)}{\lambda_2\, tanh\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right) - \lambda_1\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right)} - \frac{\lambda_1\lambda_2\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right) tanh\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right)}{\lambda_2\, tanh\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right) - \lambda_1\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right)} \tag{35}$$

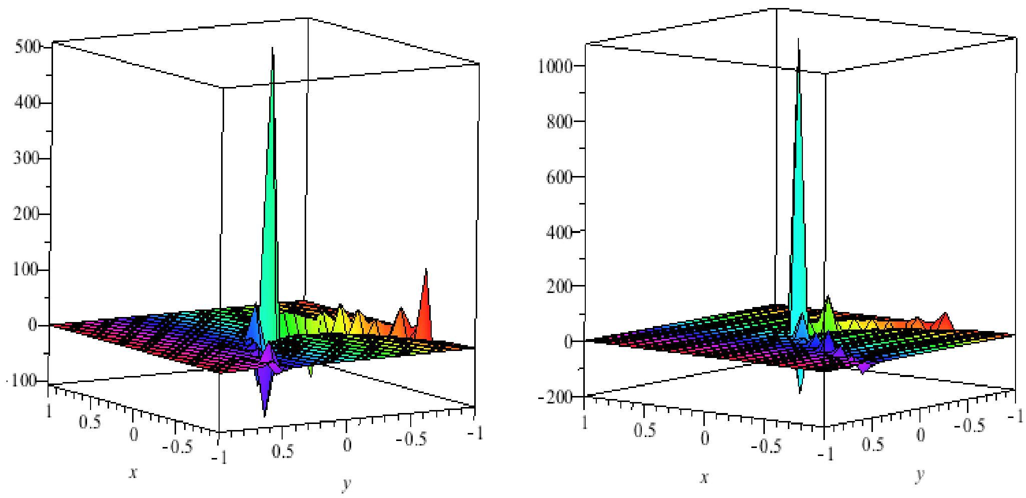
Substituting from eq. (35) and eq. (34) in eq. (26):

$$\begin{aligned} h[2] = {} & xy + \left(\frac{1}{z}\right) + \lambda_1\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right) \\ & + \frac{\lambda_2^2 - \lambda_1^2 + \lambda_2{}^2\, tanh^2\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right)}{\lambda_2\, tanh\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right) - \lambda_1\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right)} \\ & - \frac{\lambda_1\lambda_2\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right) tanh\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right)}{\lambda_2\, tanh\left(\lambda_2 x + \lambda_2{}^3 y + \frac{3}{2}\lambda_2 y^2 - \frac{1}{\lambda_2 z}\right) - \lambda_1\, tanh\left(\lambda_1 x + \lambda_1{}^3 y + \frac{3}{2}\lambda_1 y^2 - \frac{1}{\lambda_1 z}\right)} \end{aligned} \tag{36}$$

This solution is plotted in Fig. 3 for initial solution $h[0] = xy + \frac{1}{z}$.



a. h[2] for $h[2]\ for\ \lambda_1 = 0.5, \lambda_2 = 0.25, z = 5$ **b.** h[2] for $\lambda_1 = 0.5, \lambda_2 = 0.25, z = 10$



c: h[2] for $\lambda_1 = 1, \lambda_2 = 0.25, z = 5$ **d**: h[2] for $\lambda_1 = 1, \lambda_2 = 0.25, z = 10$

Fig. 3. Two solitons solution for equation (1) for different vertical distances "z "and similar λ' s

5. Conclusions

From this research, we can obtain following conclusions:

- Form Lax pair (2), (3) a new explicit solutions for Hirota Satsuma equation (1) are detected using Darboux transformation.

- Test different seed solutions and applying one and two solitons DT, a suitable choice of the seed (initial) fields leads to new solutions.
- This equation is more applicable in the flow of an incompressible fluid.
- Hirota - Satsuma has no analytical solution before except [25] and the comparison is very difficult because the solution in [25] depend on one similarity variable.

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