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Article

Meixner's Polynomial Method for Solving Integral Equations

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Abstract: This manuscript witnesses a modification in the efficient Galerkin weighted residual numerical method is proposed with Chebyshev polynomials as trial functions by inserting Meixner's polynomials instead of the traditional Chebyshev polynomials. The modified version which is called the Meixner's polynomials Method (MPM)is highly accurate and is tested on linear and nonlinear integral equations and systems. Couple of examples is given to elucidate the solution procedure. Comparison of numerical results explicitly reflects the very high level of accuracy.

Keywords: Meixner's polynomials, Galerkin weighted residual numerical method, Integral equations, MAPLE 13.

Mathematics Subject Classification Code (2010): 74S05; 42C40; 65Lxx

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1. Introduction

Integral equations [1] are useful in describing the various phenomena in disciplines. Many problems of mathematical physics can be started in the form of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. Therefore, the study of integral equations and methods for solving them are very useful in application. In recent years, there has been a growing interest in the Volterra integral equations arising in various fields of physics and engineering [2], e.g., potential theory and Dirichlet problems, electrostatics, the particle transport problems of astrophysics and reactor theory, contact problems, diffusion problems, and heat transfer problems.

$$u(x) = f(x) + \int_{\alpha(x)}^{\beta(x)} K(x,t)\, dt, \qquad (1)$$

where $\alpha(x)$ and $\beta(x)$ are function in x. Fredholm integral equations [3] arise in many scientific applications. It was also shown that Fredholm integral equations can be derived from boundary value problems. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and played a major role in the establishment of operator theory.

$$u(x) = f(x) + \int_{a}^{b} K(x,t)\, dt, \qquad (2)$$

where a and b are constants. Similarly, Abel in 1823 investigated the motion of a particle that slides down along a smooth unknown curve, in a vertical plane, under the influence of the gravity. The particle takes the time f(x) to move from the highest point of the vertical height x to the lowest point 0 on the curve. The Abel's problem is derived to find the equation of that curve. Abel's integral equation is earliest example of an integral equation [4]. Abel's integral equation has enormous application in applied problems including microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostic, X-rays radiography, fluid mechanics, bio-mechanics, electromagnetic theory and optical fiber evaluation, see [5] and references therein. The standard form of integral equation is

$$f(x) = \int_0^x \frac{u(t)}{(x^\rho - t^\rho)^\alpha}\, dt, \qquad x > 0,\, 0 < \alpha \leq 1. \qquad (3)$$

Recently, there are many approaches developed to find the exact and approximate solutions of integral equations, Yousefi et al., used CAS wavelets [6] and Legendre wavelet [7] methods to find the solutions of linear and nonlinear Fredholm integral equations, Biazar and Ebrahimi [8] implemented Chebyshev wavelets approach for nonlinear systems of Volterra integral equations, Usman et al. developed some new algorithms to seek the exact solutions of linear and nonlinear Abel's integral equation [9] and generalized Abel's integral equations [10], Moreover; In [11], Brunner et al., introduced a class of methods depending on some parameters to obtain the numerical solution of Abel integral equation of the second kind, Variational iteration method [12], Homotopy perturbation method [13-14] and Adomian's decomposition method [15] are effective and convenient for solving integral equations. Kauthen [16] applied linear multistep methods to obtain the numerical solution of a singular nonlinear Volterra integral equation. Also Kilbas and Saigo used an asymptotic method [17]to obtain numerically the solution of nonlinear Abel–Volterra integral equation. Orsi used a Product Nyström method [18], as a numerical method, to obtain the solution of nonlinear Volterra integral equation, Fettis use the Gauss–Jacobi quadrature rule [19] to determine the a numerical form of the solution of Abel equation, Huang et al. [20] used the Taylor expansion of the unknown function and obtained an approximate solution, latter on Piessens and Verbaeten [21] and Piessens [22] uses Chebyshev polynomials to developed an approximate solution to Abel equation, Yousefi uses Legendte wavelets [23] presented a numerical method for the solution of Abel integral equation. When input signal is with noisy error, Murio et al. [24] suggested a stable numerical solution. Furthermore, Garza et al. [25] and Hall et al. [26] used the wavelet method to invert the inversion of noisy Abel equation.

Inspired and motivated by the ongoing research in this area, we present a new, simple approach for solving the integral equations including Volterra integral [3], Fredholm integral [3], Integro-Differential [3], nonlinear Abel's integral [3], weakly singular [3] and system of Integral [3] equations. In the proposed scheme, we use efficient Galerkin weighted residual numerical method is proposed with Meixner's polynomials as trial functions. It is to be highlighted that suggested algorithm is extremely simple but highly effective and may be extended to other singular problems of diversified physical nature. Moreover, this new scheme is capable of reducing the computational work to a tangible level while still maintaining a very high level of accuracy.

2. Meixner's polynomials Method (MPM)

In mathematics Meixner's polynomials (also called discrete Laguerre polynomial) are a family of discrete orthogonal polynomials introduced by Josef Meixner (1934). The recurrence relation is

$$M_n(x) = \sum_k^n \frac{(-1)^k n!}{k\,!(n-k)!} \frac{x!}{k!(x-k)} \frac{k!\,\Gamma(x-\delta+1)}{\Gamma(x-\delta-n+k+1)} g^{-k} \tag{4}$$

$M_0 = 1,$

$M_1 = \frac{1}{2}x - 1,$

$$M_2 = \frac{1}{4}x^2 - \frac{9}{4}x + 2,$$

$M_3 = \frac{1}{8}x^3 - \frac{21}{8}x^2 + \frac{17}{2}x - 6,$

$$M_4 = \frac{1}{16}x^4 - \frac{19}{8}x^3 + \frac{299}{16}x^2 - \frac{323}{8}x + 24,$$

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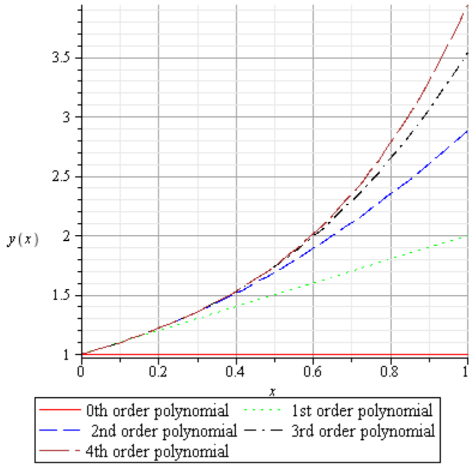


Fig. 1. Graphical representation of first five polynomials in [0,1]

3. Methodology

Integral Equation of the 1st Kind: Consider the integral equation of the 1st kind is given as

$$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt\,, a \leq x \leq b. \tag{5}$$

where $u(t)$ is the unknown function, to be determined, $\mathrm{K(x,t)}$, the kernel, is a continuous or discontinuous and square integrable function, $f(x)$ being the known function. Now we use the technique of Galerkin method [27], to find an approximate solution $\tilde{u}(x)$ of Eq. (8). For this, we assume that

$$\tilde{u}(x) = \sum_{k=0}^{n} a_k K_k(x) = \boldsymbol{U^T K(x)}, \tag{6}$$

where $U^T = [a_0, a_1, a_2, \dots]^T$, and $V(x) = [K_0(x), K_1(x), K_2(x), \dots]^T$.

where $K_k(x)$ are Kravchuk polynomials of degree k defined in Eq. (6) and a_k are unknown parameters, to be determined. Substituting Eq. (5) into Eq. (6), we get

$$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)\boldsymbol{U^T K(t)}dt, \quad a \leq x \leq b. \tag{7}$$

Then the Galerkin equations are obtained by multiplying both sides of Eq. (7) by $\boldsymbol{K}' = K_j(x), j = 0,1,2, \dots n,$ and then integrating with respect to x from a to b, we have

$$\int_a^b \boldsymbol{\phi(x)}\, dx = \int_a^b \left(\int_a^x K(x,t)\boldsymbol{K(t)K'(t)}dt \right) dx, \tag{8}$$

where $\boldsymbol{\phi(x)} = \begin{bmatrix} fK_0 \\ fK_1 \\ fK_2 \\ \vdots \\ fK_n \end{bmatrix}, \boldsymbol{K(t)K'(t)} = \begin{bmatrix} K_0K_0' & K_1K_0' & K_2K_0' \dots K_nK_0' \\ K_0K_1' & K_1K_1' & K_2K_1' \cdots K_nK_1' \\ K_0K_2' & K_1K_2' & K_2K_2' \cdots K_nK_2' \\ \vdots & \vdots & \vdots \ddots \quad \vdots \\ K_0K_n' & K_1K_n' & K_0K_n' \dots \quad K_nK_n' \end{bmatrix}.$

Since in each equation, there are two integrals. The inner integrand of the left side is a function of x, and t, and is integrated with respect to t from a to x. As a result the outer integrand becomes a function of xonly and integration with respect to x from a to b yields a constant. Thus for each $j = 0,1,2, \dots, n$ we have a linear equation with $n+1$ unknowns $a_k, k = 0,1,2, \dots, n$.

Finally the Eq. (8) can be rewrite as

$$\boldsymbol{K_{k,j}} = \boldsymbol{\chi}, \tag{9}$$

where $\boldsymbol{K_{k,j}} = \int_a^b \left(\int_a^x K(x,t)\boldsymbol{K}(\boldsymbol{t})\boldsymbol{K}'(\boldsymbol{t})dt\right)dx,\ k,j = 0,1,2,\dots n$

$$\boldsymbol{\chi} = \int_a^b f(x)\, K_j(x)dx = \int_a^b \boldsymbol{\phi}(\boldsymbol{x})\, dx, j = 0,1,2,\dots n$$

Now the unknown parameters a_k are determined by solving the equation (9) and substituting these values of parameters in Eq. (6), we get the approximate solution $\tilde{u}(x)$ of the integral equation (5).

Integral Equation of the 2nd Kind: Consider the integral equation of the 2nd kind is

$$f(x) = c(x)u(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt\,, a \le x \le b. \tag{10}$$

where $u(t)$ is the unknown function, to be determined, $k(x,t)$, the kernel, is a continuous or discontinuous and square integrable function, $f(x)$ and $u(x)$ being the known function and λ is a constant. Proceeding as before

$$\boldsymbol{K_{k,j}} = \boldsymbol{\chi}, \tag{11}$$

where $\boldsymbol{K_{k,j}} = c\int_a^b \boldsymbol{K}(\boldsymbol{t})\boldsymbol{K}'(\boldsymbol{t})dx, \int_a^b \left(\int_a^x K(x,t)\boldsymbol{K}(\boldsymbol{t})\boldsymbol{K}'(\boldsymbol{t})dt\right)dx,\ k,j = 0,1,2,\dots n$

$$\boldsymbol{\chi} = \int_a^b f(x)\, K_j(x)dx = \int_a^b \boldsymbol{\phi}(\boldsymbol{x})\, dx, j = 0,1,2,\dots n.$$

Now the unknown parameters a_k are determined by solving the equation (11) and substituting these values of parameters in Eq. (6), we get the approximate solution $\tilde{u}(x)$ of the integral equation (5).

4. Numerical Applications

In this section, we apply the proposed technique to construct approximate and analytical solutions of linear and nonlinear integral equations. Numerical results are very encouraging.

4.1. Weakly Singular Volterra Integral Equation

Consider the following weakly singular volterra Integral equation [3]

$$u(x) = x^2 - \frac{128}{14}x^{\frac{9}{4}} + \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}}u(t)\mathrm{d}t. \tag{12}$$

The exact solution of Eq. (12) is $u(x) = x^2$.

According to the proposed technique, consider the path solution

$$u(x) = \sum_{k=0}^{n} \alpha_k M_k(x). \tag{13}$$

Consider 2nd order Meixner Polynomials, i.e. for $\boldsymbol{k = 2}$, and we apply the proposed technique to solve Eq. (13) with $k = 2$. We have Eq. (13) is

$$u(x) = \sum_{k=0}^{2} \alpha_k M_k(x) = \sum_{k=0}^{2} C^T M(x), \tag{14}$$

where $C = [\alpha_0, \alpha_1, \alpha_2]^T$, and $M(x) = [M_0, M_1, M_2]^T$. Putting Eq. (14) into Eq. (12), we obtained

$$\sum_{k=0}^{2} \boldsymbol{C}^T \boldsymbol{M}_k(x) = x^2 - \frac{128}{14} x^{\frac{9}{4}} + \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}} \sum_{k=0}^{2} \boldsymbol{C}^T \boldsymbol{M}_k(t)\, \mathrm{d}t. \tag{15}$$

Multiplying both sides by $R_j(x), j = 0,1,2.$ Then integrating over [0,1] we have Eq. (18) is

$$\int_0^1 \sum_{k=0}^{2} \boldsymbol{C}^T \boldsymbol{M}_k(x) \boldsymbol{R}_j(x)\, \mathrm{d}x = \int_0^1 \left[x^2 - \frac{128}{14} x^{\frac{9}{4}}\right] \boldsymbol{R}_j(x) \mathrm{d}x$$

$$+ \int_0^1 \left[\int_0^x \frac{1}{(x-t)^{\frac{3}{4}}} \sum_{k=0}^{2} \boldsymbol{C}^T \boldsymbol{M}_k(t)\, \mathrm{d}t\right] R_j(x) \mathrm{d}x, \tag{16}$$

for $j = 0,1,2$. The matrix form of Eq. (19) is given as

$$\begin{bmatrix} -2.20 & 1.74 & -2.46 & 6.76 \\ 1.56 & -1.27 & -0.912 & -5.15 \\ -175 & 1.50 & 1.348 & 6.81 \\ -3.98 & -3.96 & 6.50 & -18.2 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -0.542 \\ 0.332 \\ -0.226 \\ 0.549 \end{bmatrix}$$

after solving we get $\alpha_0 = 10,\ \alpha_1 = 18, \alpha_2 = 4$. Consequently, we have the exact solution is $u(x) = x^2$.This is the exact solution.

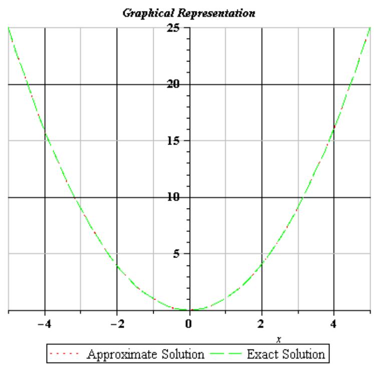


Fig. 2: Comparison of Exact and Approximate solutions of $u(x)$ of weakly singular Volterra Integral equation using Meixner polynomial of Eq. (12)

4.2. System of Weakly Singular Integral Equation

Consider the system of weakly singular integral equation of 1st Kind [3]

$$\frac{16}{195} x^{\frac{5}{4}}(32x^2 + 39) + \frac{25}{4788} x^{\frac{9}{5}}(57x^2 - 133) = \int_0^x \frac{1}{(x-t)^{\frac{3}{4}}} u(t) + \frac{1}{(x-t)^{\frac{1}{5}}} v(t) \mathrm{d}t, \tag{17a}$$

$$\frac{16}{195} x^{\frac{5}{4}}(32x^2 - 39) + \frac{25}{4788} x^{\frac{9}{5}}(57x^2 + 133) = \int_0^x \frac{1}{(x-t)^{\frac{1}{5}}} u(t) + \frac{1}{(x-t)^{\frac{3}{4}}} v(t) \mathrm{d}t. \tag{17b}$$

The exact solution of Eq. (17) is $u(x) = x^3 + x, v(x) = x^3 - x$.According to the proposed technique, consider the trail solution

$$u(x) = \sum_{k=0}^{n} \alpha_k M_k(x)\,, v(x) = \sum_{k=0}^{n} \beta_k M_k(x),$$

Table 1: Comparison of the Exact Solution and Approximate Solution of system of weakly singular integral equation using Meixner polynomial of Eq. (17) when k = 30

x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	2.0000000000	0.0000000000	2.000000000	-0.0000000000	1.7271E-25	1.3162E-26
0.1	2.1051709181	-0.1051709181	2.105179181	-0.1051709181	3.7925E-26	1.7021E-28
0.2	2.2214027582	-0.2214027582	2.221427582	-0.2214027582	8.3739E-27	1.5451E-26
0.3	2.3498588076	-0.3498588076	2.349588076	-0.3498588076	6.4059E-27	1.4846E-26
0.4	2.4918246976	-0.4918246976	2.498246976	-0.4918246976	3.4591E-26	1.0179E-26
0.5	2.6487212707	-0.6487212707	2.687212707	-0.6487212707	2.3820E-26	3.0721E-26
0.6	2.8221188004	-0.8221188004	2.822118804	-0.8221188004	1.1160E-26	9.7581E-27
0.7	3.0137527075	-1.0137527075	3.013752075	-1.0137527075	2.6344E-26	1.4123E-26
0.8	3.2255409285	-1.2255409285	3.225549285	-1.2255409285	1.3271E-26	8.2910E-29
0.9	3.4596031112	-1.4596031112	3.459631112	-1.4596031112	2.3977E-26	2.0262E-26
1.0	3.7182818285	-1.7182818285	3.718818285	-1.7182818285	2.3713E-25	3.2704E-25

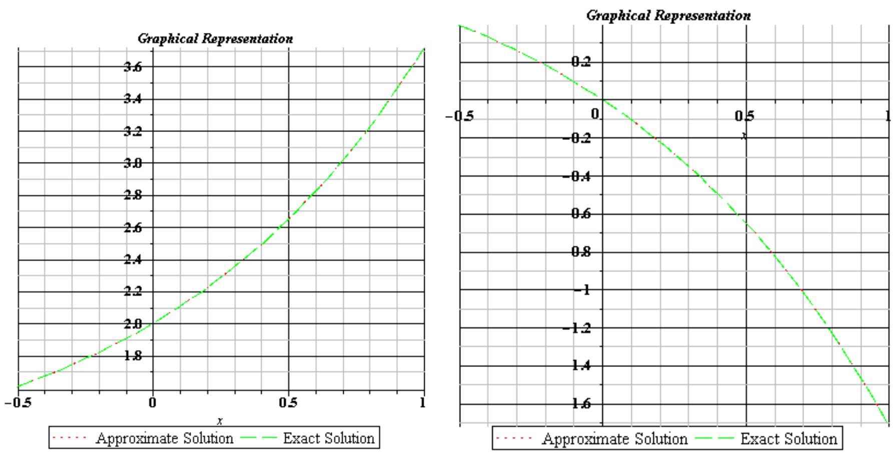


Fig. 3. (a)-(b): Comparison of Exact and Approximate solutions of $u(x)$ and $v(x)$ respectively of Volterra integral equation of 2nd Kind using Meixner's Polynomials of Eq. (17)

4.3. Linear Volterra Integral Equation

Consider the linear Volterra integral equation [3]

$$u(x) = -2 + x^2 + \sin(x) + 2\cos(x) - \int_0^x (x-t)^2 u(t)\mathrm{d}t. \quad (18)$$

The exact solution of Eq. (18) is $u(x) = (x-t)^2$.According to the proposed technique, consider the trail solution

$$u(x) = \sum_{k=0}^{n} \alpha_k M_k(x).$$

Consider 3rd order Meixner's Polynomials, i.e. for $\boldsymbol{k = 3}$ and we apply the proposed technique to solve it tables 2-4 shows the error analysis of exact and approximate solution for $\boldsymbol{k}$= 3, 20, 50 respectively.

Table 2: Comparison of the Exact Solution and Approximate Solutions of Volterra integral equation using Meixner polynomial of Eq. (18) when ***k***= 3

x	Exact Solution	Approximate Solution	Error
0.0	0.000000000000000	0.009300000000000	9.30000E-03
0.1	0.099830000000000	0.106600000000000	6.77000E-03
0.2	0.198700000000000	0.201300000000000	2.60000E-03
0.3	0.295500000000000	0.293500000000000	2.00000E-03
0.4	0.389400000000000	0.383400000000000	6.00000E-03
0.5	0.479400000000000	0.470500000000000	8.90000E-03
0.6	0.564600000000000	0.555200000000000	9.40000E-03
0.7	0.644200000000000	0.637200000000000	7.00000E-03
0.8	0.717400000000000	0.716700000000000	7.00000E-04
0.9	0.783300000000000	0.793800000000000	1.05000E-02
1.0	0.841500000000000	0.868000000000000	2.65000E-02

Table 3: Comparison of the Exact Solution and Approximate Solutions of Volterra integral equation using Meixner polynomial of Eq. (18) when ***k***= 20

x	Exact Solution	Approximate Solution	Error
0.0	0.000000000000000	-0.000000000000000	4.30445E-20
0.1	0.099833416646828	0.099833416646828	8.20511E-21
0.2	0.198669330795061	0.198669330795061	9.89853E-21
0.3	0.295520206661340	0.295520206661340	8.52020E-21
0.4	0.389418342308650	0.389418342308650	8.61156E-21
0.5	0.479425538604203	0.479425538604203	9.13538E-21
0.6	0.564642473395035	0.564642473395035	9.41831E-21
0.7	0.644217687237691	0.644217687237691	9.70310E-21
0.8	0.717356090899523	0.717356090899523	1.05778E-20
0.9	0.783326909627483	0.783326909627483	6.72317E-21
1.0	0.841470984807897	0.841470984807897	4.56533E-20

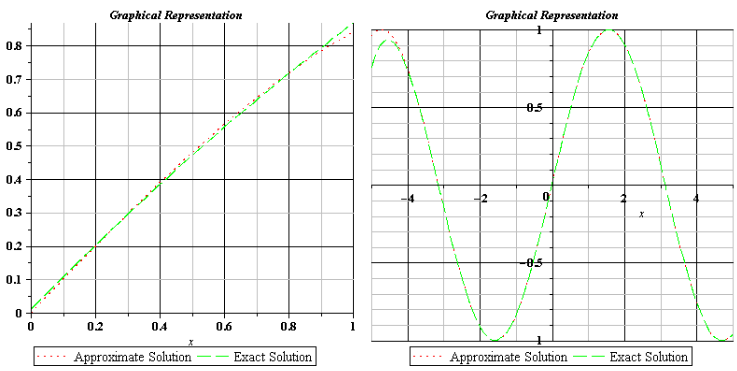


Fig. 4: Comparison of Exact and Approximate solutions of $u(x)$ of volterra integral equation using Meixner polynomial of Eq. (18)

Table 4: Comparison of the Exact Solution and Approximate Solutions of Volterra integral equation using Meixner polynomial of Eq. (18) when $\boldsymbol{k}$= 50

x	Exact Solution	Approximate Solution	Error
0.0	0.000000000000000	-0.000000000000000	4.30445E-20
0.1	0.099833416646828	0.099833416646828	8.20511E-21
0.2	0.198669330795061	0.198669330795061	9.89853E-21
0.3	0.295520206661340	0.295520206661340	8.52020E-21
0.4	0.389418342308650	0.389418342308650	8.61156E-21
0.5	0.479425538604203	0.479425538604203	9.13538E-21
0.6	0.564642473395035	0.564642473395035	9.41831E-21
0.7	0.644217687237691	0.644217687237691	9.70310E-21
0.8	0.717356090899523	0.717356090899523	1.05778E-20
0.9	0.783326909627483	0.783326909627483	6.72317E-21
1.0	0.841470984807897	0.841470984807897	4.56533E-20

4.4. System of Volterra Integral Equation

Consider the system of Volterra integral equation of 2nd kind [3]

$$u(x) = 1 - 2x + \sin(x) + \int_0^x \big(u(t) + v(t)\big), \qquad (19a)$$

$$v(x) = 1 - x^2 - \sin(x) + \int_0^x \big(tu(t) + tv(t)\big), \qquad (19b)$$

The exact solution of Eq. (22) is $u(x) = 1 + \sin(x).\, v(x) = 1 - \sin(x)$.

According to the proposed technique, consider the trail solution

$$u(x) = \sum_{k=0}^{n} \alpha_k M_k(x)\,.\, v(x) = \sum_{k=0}^{n} \beta_k M_k(x).$$

Consider 2nd order Meixner's Polynomials, i.e. for $\boldsymbol{k} = \mathbf{2}$, and we apply the proposed technique

$$u(x) = \sum_{k=0}^{2} \alpha_k M_k(x) = \sum_{k=0}^{2} \boldsymbol{C}^T \boldsymbol{M}_k(x), \quad v(x) = \sum_{k=0}^{2} \beta_k M_k(x) = \sum_{k=0}^{2} \boldsymbol{D}^T \boldsymbol{M}_k(x),$$

where $\boldsymbol{C} = [\alpha_0, \alpha_1, \alpha_2]^T, \boldsymbol{D} = [\beta_0, \beta_1, \beta_2]^T, \boldsymbol{M}(x) = [M_0, M_1, M_2]^T$.

Tables 5-7 shows the error analysis of exact and approximate solution for $\boldsymbol{k}$= 2, 20, 50 respectively.

Table 5: Comparison of Exact and Approximate solutions of $u(x)$ and $v(x)$ respectively of volterra integral equation of 2nd kind using Meixner polynomial of Eq. (22) when $\boldsymbol{k}$= 2

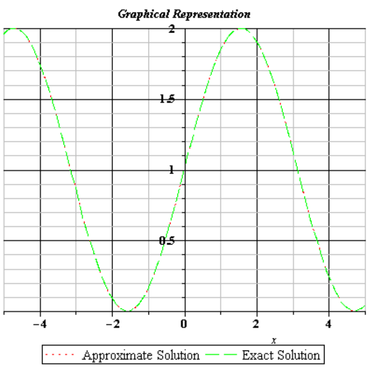
x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	1.0000000000	1.000000000	0.9340000000	0.9990000000	6.6000E-02	1.0000E-03
0.1	1.1000000000	0.900000000	1.0700000000	0.9000000000	3.0000E-02	0.0000E+00
0.2	1.2000000000	0.801000000	1.1900000000	0.8030000000	1.0000E-02	2.0000E-03
0.3	1.3000000000	0.704000000	1.3000000000	0.7100000000	0.0000E+00	6.0000E-03
0.4	1.3900000000	0.611000000	1.4100000000	0.6190000000	2.0000E-02	8.0000E-03
0.5	1.4800000000	0.521000000	1.5000000000	0.5320000000	2.0000E-02	1.1000E-02
0.6	1.5600000000	0.435000000	1.5900000000	0.4470000000	3.0000E-02	1.2000E-02
0.7	1.6400000000	0.356000000	1.6700000000	0.3660000000	3.0000E-02	1.0000E-02
0.8	1.7200000000	0.283000000	1.7300000000	0.2870000000	1.0000E-02	4.0000E-03
0.9	1.7800000000	0.21700000	1.7800000000	0.2120000000	0.0000E+00	5.0000E-03
1.0	1.8400000000	0.15900000	1.8400000000	0.1390000000	0.0000E+00	2.0000E-02

Table 6: Comparison of Exact and Approximate solutions of $u(x)$ and $v(x)$ respectively of volterra integral equation of 2nd kind using Meixner polynomial of Eq. (19) when $\boldsymbol{k}$= 20

x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	7.4752E-23	9.1892E-21
0.1	1.0998334166	0.9001665834	1.0998334166	0.9001665834	3.1510E-23	2.3426E-21
0.2	1.1986693308	0.8013306692	1.1986693308	0.8013306692	2.7038E-24	1.0677E-21
0.3	1.2955202067	0.7044797933	1.2955202067	0.7044797933	2.0801E-23	1.5343E-22
0.4	1.3894183423	0.6105816577	1.3894183423	0.6105816577	2.6255E-23	7.3522E-23
0.5	1.4794255386	0.5205744614	1.4794255386	0.5205744614	2.5710E-23	3.5118E-23
0.6	1.5646424734	0.4353575266	1.5646424734	0.4353575266	2.4325E-23	3.9200E-24
0.7	1.6442176872	0.3557823128	1.6442176872	0.3557823128	3.0840E-23	2.2004E-22
0.8	1.7173560909	0.2826439091	1.7173560909	0.2826439091	6.0265E-23	1.1173E-21
0.9	1.7833269096	0.2166730904	1.7833269096	0.2166730904	9.0111E-23	2.3182E-21
1.0	1.8414709848	0.1585290152	1.8414709848	0.1585290152	4.0095E-22	9.1256E-21

Table 7: Comparison of Exact and Approximate solutions of $u(x)$ and $v(x)$ respectively of volterra integral equation of 2nd kind using Meixner polynomial of Eq. (19) when $\boldsymbol{k}$= 50

x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	2.9524E-46	2.6971E-48
0.1	1.0998334166	0.9001665834	1.0998334166	0.9001665834	1.6024E-47	7.0444E-50
0.2	1.1986693308	0.8013306692	1.1986693308	0.8013306692	4.4196E-47	5.8193E-49
0.3	1.2955202067	0.7044797933	1.2955202067	0.7044797933	3.1770E-47	5.3425E-49
0.4	1.3894183423	0.6105816577	1.3894183423	0.6105816577	4.0162E-47	6.1982E-49
0.5	1.4794255386	0.5205744614	1.4794255386	0.5205744614	4.7749E-47	5.6567E-49
0.6	1.5646424734	0.4353575266	1.5646424734	0.4353575266	4.2471E-47	2.9415E-49
0.7	1.6442176872	0.3557823128	1.6442176872	0.3557823128	3.5261E-47	7.1455E-50
0.8	1.7173560909	0.2826439091	1.7173560909	0.2826439091	4.9288E-47	2.2560E-49
0.9	1.7833269096	0.2166730904	1.7833269096	0.2166730904	1.6410E-47	4.7603E-49
10	1.8414709848	0.1585290152	1.8414709848	0.1585290152	3.4279E-46	1.5025E-48



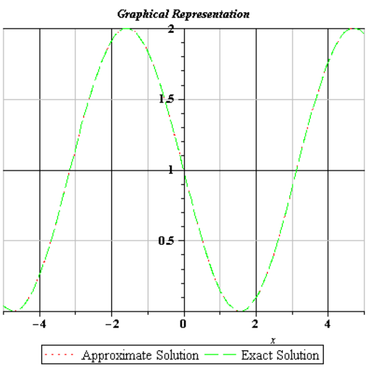


Fig. 5 (a)-(b): Comparison of Exact and Approximate solutions of $u(x)$ and $v(x)$ respectively of volterra integral equation of 2nd kind using Meixner polynomial of Eq. (19)

4.5. System of Fredholm Integral Equation

Consider the system of Fredholm integral equation [3]

$$u(x) = \left(\frac{2-\pi}{2}\right)x + x\tan^{-1}(x) + \int_{-1}^{1}(xu(t) - xv(t)\mathrm{d}t)\mathrm{d}t. \quad (20a)$$

$$v(x) = \left(\frac{3\pi-2}{6}\right) + x + \tan^{-1}(x) + \int_{-1}^{1}(tu(t) - tv(t))\mathrm{d}t. \quad (20b)$$

The exact solution of Eq. (23) is $u(x) = x\tan^{-1}(x)\,, v(x) = x + \tan^{-1}(x)$.According to the proposed technique, table 8-10 shows the error analysis of exact and approximate solution for $\boldsymbol{k}$= 2, 25, 50 respectively.

Table 8: Comparison of Exact and Approximate solutions of $u(x)$ and $v(x)$ respectively of volterra integral equation of 2nd kind using Meixner polynomial of Eq. (20) when $\boldsymbol{k}$= 2

x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	1.0000000000	1.000000000	0.9340000000	0.9990000000	6.6000E-02	1.0000E-03
0.1	1.1000000000	0.900000000	1.0700000000	0.9000000000	3.0000E-02	0.0000E+00
0.2	1.2000000000	0.801000000	1.1900000000	0.8030000000	1.0000E-02	2.0000E-03
0.3	1.3000000000	0.704000000	1.3000000000	0.7100000000	0.0000E+00	6.0000E-03
0.4	1.3900000000	0.611000000	1.4100000000	0.6190000000	2.0000E-02	8.0000E-03
0.5	1.4800000000	0.521000000	1.5000000000	0.5320000000	2.0000E-02	1.1000E-02
0.6	1.5600000000	0.435000000	1.5900000000	0.4470000000	3.0000E-02	1.2000E-02
0.7	1.6400000000	0.356000000	1.6700000000	0.3660000000	3.0000E-02	1.0000E-02
0.8	1.7200000000	0.283000000	1.7300000000	0.2870000000	1.0000E-02	4.0000E-03
0.9	1.7800000000	0.21700000	1.7800000000	0.2120000000	0.0000E+00	5.0000E-03
1.0	1.8400000000	0.15900000	1.8400000000	0.1390000000	0.0000E+00	2.0000E-02

Table 9: Comparison of the Exact Solution and Approximate Solution of Fredholm integral equation of 2nd Kind using Meixner polynomial of Eq. (20) when $\boldsymbol{k}$= 25

x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	0.0000000000	0.0000000000	0.0000000215	-0.000000003	2.1535E-08	3.4755E-10
0.1	0.0099668652	0.1996686525	0.0099668613	0.1996686207	3.9236E-09	3.1755E-08
0.2	0.0394791120	0.3973955598	0.0394790929	0.3973955744	1.9030E-08	1.4557E-08
0.3	0.0874370383	0.5914567945	0.0874370516	0.5914568185	1.3290E-08	2.4014E-08
0.4	0.1522025508	0.7805063771	0.1522025604	0.7805063491	9.5783E-09	2.7976E-08
0.5	0.2318238045	0.9636476090	0.2318237864	0.9636476057	1.8116E-08	3.3478E-09
0.6	0.3242517002	1.1404195003	0.3242517076	1.1404195294	7.4603E-09	2.9169E-08
0.7	0.4275081751	1.3107259644	0.4275081804	1.3107259343	5.3322E-09	3.0115E-08
0.8	0.5397927538	1.4747409422	0.5397927439	1.4747409636	9.8871E-09	2.1408E-08
0.9	0.6595335916	1.6328151018	0.6595335976	1.6328150723	6.0141E-09	2.9489E-08
1.0	0.7853981634	1.7853981634	0.7853981559	1.7853980371	7.4669E-09	1.2629E-07

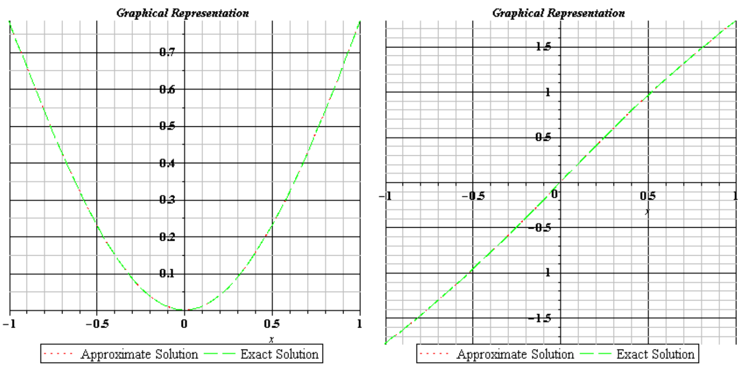


Fig. 6 (a)-(b): Comparison of Exact Approximate solutions of $u(x)$ and $v(x)$ respectively of Fredholm integral equation of 2nd Kind using Meixner polynomial of Eq. (20)

Table 10: Comparison of the Exact Solution and Approximate Solution of Fredholm integral equation of 2nd Kind using Meixner polynomial of Eq. (20) when $\boldsymbol{k}$= 50

x	u(x)	v(x)	u(x)_approx.	v(x)_approx.	Error in u(x)	Error in v(x)
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	3.2846E-14	4.5468E-15
0.1	0.0099668652	0.1996686525	0.0099668652	0.1996686525	3.2364E-14	1.2963E-15
0.2	0.0394791120	0.3973955598	0.0394791120	0.3973955598	3.0615E-14	2.1453E-15
0.3	0.0874370383	0.5914567945	0.0874370383	0.5914567945	2.6503E-14	5.7789E-15
0.4	0.1522025508	0.7805063771	0.1522025508	0.7805063771	1.7855E-14	9.0715E-15
0.5	0.2318238045	0.9636476090	0.2318238045	0.9636476090	1.6143E-15	1.0244E-14
0.6	0.3242517002	1.1404195003	0.3242517002	1.1404195003	2.2058E-14	5.7631E-15
0.7	0.4275081751	1.3107259644	0.4275081751	1.3107259644	3.3121E-14	5.4400E-15
0.8	0.5397927538	1.4747409422	0.5397927538	1.4747409422	1.6973E-14	4.3353E-15
0.9	0.6595335916	1.6328151018	0.6595335916	1.6328151018	1.6627E-14	2.7885E-15
1.0	0.7853981634	1.7853981634	0.7853981634	1.7853981634	1.8494E-13	3.5721E-14

4.6. Mixed Fredholm-Volterra Integral Equation

Consider the Mixed Volterra-Fredholm integral equation [3]

$$u(x) = -2x + x\cos(x) + \int_0^x tu(t)\mathrm{d}t + \int_0^\pi xu(t)\mathrm{d}t, \tag{21}$$

The exact solution of Eq. (24) is $u(x) = \sin(x)$. According to the proposed technique, table 11-12 shows the error analysis of exact and approximate solution for $\boldsymbol{k}$= 3, 30 respectively.

Table 11: Comparison of the Exact Solution and Approximate Solution of Volterra-Fredholm integral equation using Meixner polynomial of Eq. (24)

x	Exact Solution	Approximate Solution	Error
0.0	0.000000000000000	-0.080300000000000	8.03000E-02
0.1	0.099830000000000	0.059390000000000	4.04400E-02
0.2	0.198700000000000	0.188600000000000	1.01000E-02
0.3	0.295500000000000	0.307400000000000	1.19000E-02
0.4	0.389400000000000	0.416000000000000	2.66000E-02
0.5	0.479400000000000	0.514500000000000	3.51000E-02
0.6	0.564600000000000	0.603000000000000	3.84000E-02
0.7	0.644200000000000	0.681900000000000	3.77000E-02
0.8	0.717400000000000	0.751500000000000	3.41000E-02
0.9	0.783300000000000	0.811100000000000	2.78000E-02
1.0	0.841500000000000	0.861500000000000	2.00000E-02

Table 12: Comparison of the Exact Solution and Approximate Solution of Volterra-Fredholm integral equation using Meixner polynomial of Eq. (21)

x	Exact Solution	Approximate Solution	Error
0.0	0.000000000000000	-0.000000000000000	3.70647E-17
0.1	0.099833416646828	0.099833416646828	1.08453E-17
0.2	0.198669330795061	0.198669330795061	7.84194E-18
0.3	0.295520206661340	0.295520206661340	1.24330E-20
0.4	0.389418342308650	0.389418342308650	7.99543E-18
0.5	0.479425538604203	0.479425538604203	3.53150E-18
0.6	0.564642473395035	0.564642473395035	7.01802E-18
0.7	0.644217687237691	0.644217687237691	2.51643E-18
0.8	0.717356090899523	0.717356090899523	7.28613E-18
0.9	0.783326909627483	0.783326909627483	5.89496E-19
1.0	0.841470984807897	0.841470984807897	6.63256E-18

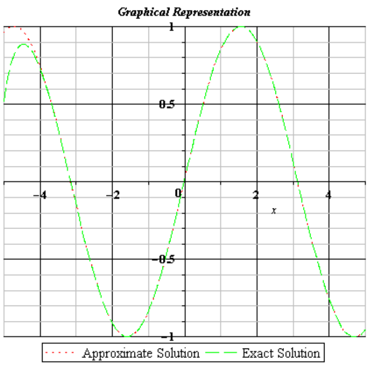


Fig. 7: Comparison of Approximate and exact solution of Volterra-Fredholm integral equation using Meixner polynomial of Eq. (21)

4.7. Nonlinear Abel's Integral Equation

Consider the nonlinear Abel's integral equation [3]

$$\frac{3}{40}x^{\frac{2}{3}}(20-24x+9x^2)=\int_0^x \frac{1}{(x-t)^{\frac{1}{3}}}u^2(\mathrm{t})\mathrm{d}t. \tag{22}$$

The exact solution of Eq. (22) is $u(x)=(1-\mathrm{x})$.According to the proposed technique, we get the exact solution for $k=2$.

5. Conclusions

A proposed technique based on Galerkin weighted residual numerical method is proposed with Modified Lommel's polynomials is developed and applied to obtain exact and approximate solutions of linear and nonlinear integral equations. Table 1-3 and Figure 2 and 3 shows the efficiency of the proposed technique, as we increase n, i.e., order of polynomial solution have less error. The proposed algorithm is extremely simple, highly effective and is of utmost accuracy.

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