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Article

A Study of Generalized Fractional Derivative Operator Involving Aleph $(\aleph)$-function, General Class of Multivariable Polynomials and Certain Multiplication Formulae for Aleph $(\aleph)$-Function of Multivariable Functions

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Abstract: The aim of the present paper is to create two new and useful theorems by using the generalized differential operator $D^{m}_{k,\alpha,x}$ involving a general multivariable polynomial and Aleph $(\aleph)$-function. And one theorem creates by Generalized Fractional derivative Operator involving $D^{\alpha,\beta,\eta}_{0,x,m}$ involving a general multivariable polynomial and multivariable Aleph $(\aleph)$-function. Two Multivariable formulae for Aleph function of multivariable functions have been obtained as special cases of our main result.

Keywords: Generalized Hypergeometric function, Aleph $(\aleph)$-function, Fractional differential operator, General class of multivariable polynomials and generalized Lauricella function.

Mathematics Subject Classification 2010: 33EXX, 33E30, 31A10, 33C05, 33C60, 33C70, 26A33

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1. Introduction

THE GENERAL MULTIVARIABLE POLYNOMIALS

The Multivariable polynomial, $S_V^{U_1,\ldots,U_k}(x_1,\ldots,x_k)$ introduced by Srivastava and Garg (1987) [15, p.686, eq. (1.4)] is defined in the following manner:

$$S_V^{U_1,\ldots,U_k}(x_1,\ldots,x_k)=\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k}U_iR_i\le V}}(-V)_{\sum_{i=1}^{k}U_iR_i}\,A(V,R_1,\ldots,R_k)\,\frac{x_i^{R_i}}{R_i!} \tag{1}$$

Where $V=0,1,2\ldots$ and $U_1,\ldots,U_k$ arbitrary positive integers and the coefficients are $A(V,R_1,\ldots,R_k)$ is arbitrary constants (real or complex).Evidently the case k=1 of the Polynomial (1) would correspond to the Polynomial $S_V^U[x]$ given by Srivastava [12].

ALEPH ($\aleph$)-FUNCTION

Sudland [17] Introduced the Aleph ($\aleph$) function, however the notation and complete definition is acquainted here in the following way in terms and the Mellin- Barnes type integrals Barnes type integrals

$$\aleph[z]=\aleph_{P_i,Q_i;\tau_i;r}^{M,N}\left[z\left|\begin{matrix}(a_j,A_j)_{1,N},\left[\tau_i(a_{ji},A_{ji})\right]_{N+1,P_i}\\ (b_j,B_j)_{1,M},\left[\tau_i(b_{ji},B_{ji})\right]_{M+1,Q_i}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int_L\Omega_{P_i,Q_i;\tau_i;r}^{M,N}(s)z^{-s}ds \tag{2}$$

For all $z\neq 0$ where $\omega=\sqrt{(-1)}$ and

$$\Omega_{P_i,Q_i;\tau_i;r}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{N}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\left(\prod_{j=M+1}^{Q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)\prod_{j=N+1}^{P_i}\Gamma\left(a_{ji}+A_{ji}s\right)\right)} \tag{3}$$

The integration path $L=L_{i\gamma\infty},\gamma\in R$ extends from $\gamma-i\infty$ to $\gamma+i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1-a_j-A_js)$, $j=i,\ldots,N$ do not coincide with the pole of $\Gamma(b_j+B_js)$, $j=i,\ldots,M$ the parameter P_i,Q_i are non-negative integers satisfying: $0\le N\le P_i, 0\le M\le Q_i, \tau_i>0$ for $i=1,\ldots,r$. The

$A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The empty product in (3) is interpreted as unity. The existence conditions for the defining integral (2) are as following:

$$\text{(i) } \phi_l > 0, |\arg(z)| < \frac{\pi}{2}\phi_l, \; l = 1,2,\ldots r \tag{4}$$

$$\text{(ii) } \phi_l \geq 0, |\arg(z)| < \frac{\pi}{2}\phi_l, \; R(\xi_l) + 1 < 0, \tag{5}$$

Where

$$\phi_l = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \tau_l \left(\sum_{j=N+1}^{P_l} A_{jl} + \sum_{j=M+1}^{Q_l} B_{jl} \right) \tag{6}$$

$$\xi_l = \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j + \tau_l \left(\sum_{j=M+1}^{Q_l} b_{jl} - \sum_{j=N+1}^{P_l} a_{jl} \right) + \frac{1}{2}(P_l - Q_l), \forall l = 1,2,\ldots,r \tag{7}$$

Detailed introduction of Aleph ($\aleph$) function is given in [17] and [18].

MULTIVARIABLE ALEPH $\aleph$ - FUNCTIONS

The function Aleph of several variables generalize the multivariable I-function recently study by Sharma and Ahmad [11] and Ayant [1], itself is a generalization of G and H-function of multiple variables. The multiple Mellin-Barnes integral occurring in this thesis will be referred to as the multivariable's Aleph-function throughout our present study and will be defined and represented as follow.

We have

$$\aleph[z_1,\ldots,z_r] = \aleph^{0,n;m_1,n_1;\ldots;m_r,n_r}_{p_i,q_i,\tau_i;R:p_i(1),q_i(1),\tau_i(1);R^{(1)};\ldots;p_i(r),q_i(r),\tau_i(r);R^{(r)}} \left[\begin{array}{c|c} z_1 & A^{**} \\ z_r & B^{**} \end{array} \right]$$

$$A^{**} = \left[\left(a_j, \alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}\right], \left[\tau_i\left(a_{ji}, \alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)\right]_{n+1,p_i}; \left[\left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,n_1}\right],$$

$$\left[\tau_{i(1)}\left(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}\right)\right]_{n_1+1,p_i^{(1)}}; \ldots; \left[\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r}\right], \left[\tau_{i(r)}\left(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}\right)\right]_{n_r+1,p_i^{(r)}}$$

$$B^{**} = \quad , \left[\tau_i\left(b_{ji}, \beta_j^{(1)},\ldots,\beta_j^{(r)}\right)\right]_{m+1,q_i}; \left[\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,n_1}\right],$$

$$\left[\tau_{i(1)}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}}; \ldots; \left[\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}\right], \left[\tau_{i(r)}\left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}}$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\ldots,s_r) \prod_{k=1}^{r} \theta(s_k)\, z_k^{s_k}\, ds_1 \ldots ds_r \tag{8}$$

Where $\omega = \sqrt{(-1)}$

$$\psi(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{n} \Gamma\left(1-a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k\right)}{\sum_{i=1}^{R}\left[\tau_i \prod_{j=n+1}^{p_i} \Gamma\left(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k\right) \prod_{j=1}^{q_i} \Gamma\left(1-b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k\right)\right]}$$

$$\theta(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) \prod_{j=1}^{n_k} \Gamma\left(1-c_j^{(k)} + \gamma_j^{(k)} s_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma\left(1-d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k\right) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma\left(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k\right)\right]}$$

Where $j = 1,\ldots,r$ and $k = 1,\ldots,r$

Fractional Differential Operator

Two types of fractional operators have been studied.

(i) Riemann-Liouville Fractional Operator

Oldham and spanier [9] defined the Riemann-Liouville Fractional Operator for function $f(z)$ of a complex order μ can be represented in the following manner:

$${}_{\alpha}D_x^{\mu} f(x) = \begin{cases} \frac{1}{\Gamma(-\mu)} \int_{\alpha}^{x} (x-t)^{-\mu-1} f(t)\,dt &, \mathrm{Re}(\mu) < 0 \\ \frac{d^m}{dx^m}\, {}_{\alpha}D_x^{\mu-m} f(x), & 0 \le \mathrm{Re}(\mu) < m \end{cases} \tag{9}$$

Where is m non -negative integral.

Mishra [8] further generalized the above operator as follow:

$$D_{k,\alpha,x} = x^{k+\alpha} D_x^{\alpha}, \ \alpha \neq \mu$$

(ii) Generalized Saigo Fractional Derivative Operator

Generalized modified fractional derivative operator due to Saigo [16] is defined as given below, if we let $0 \le \alpha < 1, \beta, \eta, x \in \Re, m \in N$.

$$D_{0,x,m}^{\alpha,\beta,\eta} f(x) = \frac{d}{dx}\left(\frac{x^{m(\beta-\eta)}}{\Gamma(1-\alpha)} \int_0^x (x^m - t^m)^{-\alpha} \; {}_2F_1\left[\begin{matrix} \beta-\alpha; 1-\eta \\ 1-\alpha \end{matrix} ; 1-\frac{t^m}{x^m} \right] f(t) dt^m \right) \tag{10}$$

The above operator reduces to Saigo derivative operator $D_{0,x}^{\alpha,\beta,\eta}$ when m=1.

On putting $\alpha = \beta$ and m=1, in the operator defined by equation (10), it reduces to the well-known Riemann-Liouville Fractional derivative Operator defined by Miller and Ross [7]. We have

$$D_{0,x}^{\alpha,\beta,\eta} f(x) = D_x^{\alpha} f(x)$$

SOME RESULTS

$${}_{\alpha}D_x^{\mu}\left(x^{\mu-1} \right) = \frac{d^{\alpha} x^{\mu-1}}{dx^{\alpha}} = \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)} x^{\mu-\alpha-1}, \alpha \neq \mu \tag{11}$$

$$D_{k,\alpha,x}^{\mu}\left(x^{\mu} \right) = \prod_{p=0}^{m-1} \frac{\Gamma(\mu + pk + 1)}{\Gamma(\mu + pk + 1 - \alpha)} x^{\mu+km}, \alpha \neq \mu + 1 \tag{12}$$

Where α and k are not necessarily integers.

In establishing our Theorem 3, the following result which is given by Bhatt and Raina [2] will also be required.

If $0 \le \alpha < 1, m \in N; \beta, \eta, x \in \Re; k > \max\{0, m(\beta-\eta)\} - m$, then

$$D_{0,x,m}^{\alpha,\beta,\eta}\left(x^{k} \right) = \frac{\Gamma\left(1+\frac{k}{m}\right)\Gamma\left(1+\eta-\beta+\frac{k}{m}\right)}{\Gamma\left(1-\beta+\frac{k}{m}\right)\Gamma\left(1+\eta-\alpha+\frac{k}{m}\right)} x^{k-m\beta} \tag{13}$$

2. Main Theorems

Theorem 1

$$D_{l,\lambda-\mu,t}^{m}\left\{ t^{\lambda-1} S_V^{U_1,\dots,U_k}\left\{ w_1 t^{\rho_1},\dots,w_k t^{\rho_k} \right\} \aleph(zt)\, f(xt) \right\}$$

$$= \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V,R_1,\ldots,R_k) \frac{w_1^{R_1}\ldots w_k^{R_k}}{R_1!\ldots R_k!}$$

$$\times \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\rho_1R_1+\ldots+\rho_kR_k-s+pl)}{\Gamma(\mu+\rho_1R_1+\ldots+\rho_kR_k-s+pl)}$$

$$\times t^{\lambda+h+\rho_1R_1+\ldots+\rho_kR_k+ml-s-1} \left\{\frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)\, z^{-s} ds\right\} {}_{m+1}F_m\left\{\begin{matrix}-n, A^* \\ B^*\end{matrix}; t\right\} D_x^n \{f(x)\} \tag{14}$$

$$A^* = \lambda+\rho_1R_1+\ldots+\rho_kR_k-s, \lambda+\rho_1R_1+\ldots+\rho_kR_k-s+l,\ldots,$$
$$\lambda+\rho_1R_1+\ldots+\rho_kR_k-s+(m-1)l$$

$$B^* = \mu+\rho_1R_1+\ldots+\rho_kR_k-s, \mu+\rho_1R_1+\ldots+\rho_kR_k-s+l,\ldots,$$
$$\mu+\rho_1R_1+\ldots+\rho_kR_k-s+(m-1)l$$

Theorem 2

$$D^m_{l,\lambda-\mu,t}\left\{ t^{\lambda} S_V^{U_1,\ldots,U_k}\left\{w_1 t^{\rho_1},\ldots,w_k t^{\rho_k}\right\} \aleph(zt) f(xt)\right\}$$

$$= \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V,R_1,\ldots,R_k) \frac{w_i^{R_i}}{R_i!} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}$$

$$\times \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\rho_1R_1+\ldots+\rho_kR_k-s+pl)(1-\mu-\rho_1R_1-\ldots-\rho_kR_k+s-pl)_n}{\Gamma(\mu+\rho_1R_1+\ldots+\rho_kR_k-s+pl)(1-\lambda-\rho_1R_1-\ldots-\rho_kR_k+s-pl)_n}$$

$$\times t^{\lambda+\rho_1R_1+\ldots+\rho_kR_k+ml-s-1} \left\{\frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s)\, z^{-s} ds\right\} {}_{m+1}F_m\left\{\begin{matrix}-n, C^* \\ D^*\end{matrix}; t\right\} D_x^n \{x^n f(x)\} \tag{15}$$

$$C^* = \lambda+\rho_1R_1+\ldots+\rho_kR_k-s-n, \lambda+\rho_1R_1+\ldots+\rho_kR_k-s-n+l,\ldots,$$
$$\lambda+\rho_1R_1+\ldots+\rho_kR_k-s-n+(m-1)l$$

$$D^* = \mu+\rho_1R_1+\ldots+\rho_kR_k-s-n, \mu+\rho_1R_1+\ldots+\rho_kR_k-s-n+l,\ldots,$$
$$\mu+\rho_1R_1+\ldots+\rho_kR_k-s-n+(m-1)l$$

The Theorems 1 and 2 are valid under the following (sufficient) conditions:

$|t|<1, \rho_i > 0 \ (i=0,1,\ldots,k), \mathrm{Re}\left(\mu + \sum_{i=0}^{k} \rho_i R_i - n + pl\right) > 0,$

and $\mathrm{Re}\left(\lambda + \sum_{i=0}^{k} \rho_i R_i + pl\right) > 0, \ (p = 0,1,\ldots,m-1)$

and by assuming that the series involving in (14) and (15) are absolutely convergent.

Proof of Theorem 1

Let us consider the well-known Taylor's expansion

$$f(xt) = \sum_{n=0}^{\infty} \frac{(t-1)^n}{n!} x^n D_x^n \{f(x)\} \tag{16}$$

This is a particular case of Lagrange's expansion.

Now multiplying both side of (16) by $t^{\lambda-1} S_V^{U_1,\ldots,U_k}\left\{w_1 t^{\rho_1},\ldots,w_k t^{\rho_k}\right\} \aleph(zt)$ and apply the operator $D^m_{l,\lambda-\mu,t}$ both sides, we get:

$$D^m_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} S_V^{U_1,\ldots,U_k}\left\{w_1 t^{\rho_1},\ldots,w_k t^{\rho_k}\right\} \aleph(zt) f(xt)\right\}$$

$$= \sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k} U_i R_i \le V} (-V)_{\sum_{i=1}^{k} U_i R_i} A(V, R_1,\ldots,R_k)$$

$$\times \frac{w_1^{R_1} \ldots w_k^{R_k}}{R_1! \ldots R_k!} \sum_{n=0}^{\infty} \sum_{h=0}^{n} \frac{(-1)^n (-n)_h \, x^n}{h! \, n!}$$

$$\times D^m_{l,\lambda-\mu,t}\left\{ t^{\lambda+h+\rho_1 R_1+\ldots+\rho_k R_k+ml-s-1}\right\} D_x^n \{f(x)\} \tag{17}$$

Now using (12) in R.H.S. of (17), the desired result can be easily achieved by a little simplification.

Similarly we prove Theorem 2 by using following expansion (17),

$$t\, f(xt) = \sum_{n=0}^{\infty} \frac{\left(1-\frac{1}{t}\right)^n}{n!} D_x^n \left\{x^n f(x)\right\}$$

Theorem-3

Let $0 \le \alpha < 1, \beta, \eta, x \in \Re, \theta \in N, \mathrm{Re}(\alpha) > 0, w, \delta > 0, \rho_j, \sigma_j > 0 \ (j=1,\ldots,k), \lambda, \mu > 0, z \in C.$ If the existence conditions of Aleph ($\aleph$) function are satisfied, then the generalized fractional derivative $D^{\alpha,\beta,\eta}_{0,x,\theta}$ of the product of Multivariable Aleph ($\aleph$) function and $S_V^{U_1,\ldots,U_k}$ exists and we have,

$$D_{0,x,\theta}^{\alpha,\beta,\eta}\left[t^{w}\left(t^{\nu}+\xi^{\nu}\right)^{-\delta}S_{V}^{U_1,\ldots,U_k}\left\{Y_1 t^{\rho_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_1},\ldots,Y_k t^{\rho_k}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_k}\right\}\right.$$

$$\left.\times\aleph\left(z_1 t^{\lambda_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_1},\ldots,z_p t^{\lambda_p}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_p}\right)\right](x)$$

$$=x^{w-\theta\beta+\sum_{i=1}^{k}\rho_i R_i}\,\xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_i R_i}\sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k}U_i R_i\le V}(-V)_{\sum_{i=1}^{k}U_i R_i}$$

$$\times A\left(V,R_1,\ldots,R_k\right)\frac{Y_1^{R_1}\ldots Y_k^{R_k}}{R_1!\ldots R_k!}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-x^{\nu}}{\xi^{\nu}}\right)^{m}$$

$$\times\aleph_{P_i+3,Q_i+3,\tau_i,R;U_{11}}^{0,N+3;m_1,n_1;\ldots;m_p,n_p}\left[\begin{array}{c}z_1\xi^{-\mu_1\nu}x^{\lambda_1}\\ \vdots\\ z_p\xi^{-\mu_p\nu}x^{\lambda_p}\end{array}\middle|\begin{array}{c}E^{*}\\ F^{*}\end{array}\right] \tag{18}$$

Where

$$U_{11}=P_i(1),Q_i(1),\tau_i(1);R^{(1)};\ldots;P_i(r),Q_i(r),\tau_i(r);R^{(r)}$$

$$E^{*}=\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n},\left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\ldots,\mu_p\right),\left(\frac{-w-m\nu-\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left(\beta-\eta-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left[\tau_i\left(a_{ji},\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)\right]_{n+1,P_i};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1},$$

$$\left[\tau_{i(1)}\left(c_{ji(1)}^{(1)},\gamma_{ji(1)}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r},\left[\tau_{i(r)}\left(c_{ji(r)}^{(r)},\gamma_{ji(r)}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

$$F^{*}=\left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\ldots\mu_p\right),\left(\beta-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left(\alpha-\eta-\frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left[\tau_i\left(b_{ji},\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)\right]_{m+1,Q_i};\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i(1)}\left(d_{ji(1)}^{(1)},\delta_{ji(1)}^{(1)}\right)\right]_{m_1+1,Q_i^{(1)}}$$

$$;\ldots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i(r)}\left(d_{ji(r)}^{(r)},\delta_{ji(r)}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}}$$

Proof:

$$D_{0,x,\theta}^{\alpha,\beta,\eta}\left[t^{w}\left(t^{\nu}+\xi^{\nu}\right)^{-\delta}\sum_{R_1,\dots,R_k=0}^{\sum_{i=1}^{k}U_iR_i\le V}\frac{(-V)_{\sum_{i=1}^{V}U_iR_i}}{}A\left(V,R_1,\dots,R_k\right)\frac{Y_1^{R_1}\dots Y_k^{R_k}}{R_1!\dots R_k!}\right.$$

$$\times\left(t^{\nu}+\xi^{\nu}\right)^{-\sum_{i=1}^{k}\sigma_iR_i}t^{\sum_{i=1}^{k}\rho_iR_i}\frac{1}{(2\pi i)^p}\int_{L_1}\dots\int_{L_p}\psi\left(s_1..s_p\right)\phi_1\left(s_1\right)\dots\phi_p\left(s_p\right)$$

$$\left.\times z_1^{s_1}\dots z_p^{s_p}\;t^{\sum_{j=1}^{p}\lambda_js_j}\left(t^{\nu}+\xi^{\nu}\right)^{-\sum_{j=1}^{p}\mu_js_j}ds_1..ds_p\right](x) \tag{19}$$

Interchanging the order of integration and summation (which is justified under the conditions, given above) and expanding the binomial term like,

$$\left(t+\xi\right)^{-l}=\xi^{-l}\sum_{m=0}^{\infty}\frac{(l)_m}{m!}\left(-\frac{t}{\xi}\right)^m\quad\left|\frac{t}{\xi}\right|<1 \tag{20}$$

The right hand side of (19) gets the following form:

$$\sum_{R_1,\dots,R_k=0}^{\sum_{i=1}^{k}U_iR_i\le V}(-V)_{\sum_{i=1}^{V}U_iR_i}A\left(V,R_1,\dots,R_k\right)\frac{Y_1^{R_1}\dots Y_k^{R_k}}{R_1!\dots R_k!}$$

$$\times\frac{1}{(2\pi i)^p}\int_{L_1}\dots\int_{L_p}\psi\left(s_1\dots s_p\right)\varphi_1\left(s_1\right)\dots\varphi_p\left(s_p\right)z_1^{s_1}\dots z_p^{s_p}$$

$$\times t^{\sum_{j=1}^{p}\lambda_js_j}\left(t^{\nu}+\xi^{\nu}\right)^{-\left(\delta+\sum_{i=1}^{k}\sigma_iR_i+\sum_{j=1}^{p}\mu_js_j\right)}$$

$$\times\xi^{-\nu\left(\delta+\sum_{i=1}^{k}\sigma_iR_i+\sum_{j=1}^{p}\mu_js_j\right)}\sum_{m=0}^{\infty}\frac{\left(\delta+\sum_{i=1}^{k}\sigma_iR_i+\sum_{j=1}^{p}\mu_js_j\right)_m}{m!}\left(-\frac{1}{\xi^{\nu}}\right)^m$$

$$\times D_{0,x,\theta}^{\alpha,\beta,\eta}\left\{t^{w+\sum_{i=1}^{k}\rho_iR_i+\sum_{j=1}^{p}\lambda_js_j+\nu m}\right\}(x)$$

The required result can be obtained by using the result (13) and putting a little simplification.

3. Special Cases of Theorems

(i) If in Theorem 1, we reduce the general class of polynomial to the first class of hypergeomeric polynomials $F_D^{(k)}$ in [16] we get the following corollary:

Corollary 1

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} F_D^{(k)}\left\{\left(-V, U_i\right);\left(\beta_i,\phi_i\right);\left(\gamma,\psi_i\right); w_1 t^{\rho_1},\dots,w_k t^{\rho_k}\right\} \aleph(zt) f(xt)\right\}$$

$$= \sum_{\substack{R_1,R_2,\dots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{V} U_i R_i} \frac{w_1^{R_1}\dots w_k^{R_k}}{R_1!\dots R_k!} \frac{(\beta_1)_{R_1\phi_1}\cdots(\beta_k)_{R_k\phi_k}}{(\gamma)_{R_1\psi_1+\dots+R_k\psi_k}} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$\times \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\rho_1 R_1+\dots+\rho_k R_k - s + pl)}{\Gamma(\mu+\rho_1 R_1+\dots+\rho_k R_k - s + pl)}\, t^{\lambda+\rho_1 R_1+\dots+\rho_k R_k + ml - s - 1}$$

$$\times \left\{\frac{1}{2\pi i}\int_L \Omega^{M,N}_{P_i,Q_i,\tau_i;r}(s) z^{-s} ds\right\}\ {}_{m+1}F_m\left\{\begin{matrix}-n, G^* \\ H^*\end{matrix}; t\right\} D_x^n\{f(x)\} \tag{21}$$

$$G^* = \lambda+\rho_1 R_1+\dots+\rho_k R_k - s, \lambda+\rho_1 R_1+\dots+\rho_k R_k - s + l,\dots,$$
$$\lambda+\rho_1 R_1+\dots+\rho_k R_k - s + (m-1)l$$

$$H^* \mu+\rho_1 R_1+\dots+\rho_k R_k - s, \mu+\rho_1 R_1+\dots+\rho_k R_k - s + l,\dots,$$
$$\mu+\rho_1 R_1+\dots+\rho_k R_k - s + (m-1)l$$

The conditions of validity of (21) can be easily obtained from the existence conditions mentioned with theorem 1.

(ii) If we reduce the general class of polynomial $S_V^{U_1,\dots,U_k}$ involved in theorem 1 to Multivariable Jacobi Polynomial $P_V^{\alpha_1,\beta_1;\dots;\alpha_k,\beta_k}$ given by Srivastava [13], the following result can be seen:

Corollary 2

$$D^{m}_{l,\lambda-\mu,t}\left\{ t^{\lambda-1} P_V^{\alpha_1,\beta_1;\dots;\alpha_k,\beta_k}\left(1-2w_1 t^{\rho_1},\dots,1-2w_k t^{\rho_k}\right) \aleph(zt) f(xt)\right\}$$

$$= \frac{\prod_{i=1}^{k}\left(1+\alpha_i\right)_V}{(V!)^k} \sum_{\substack{R_1,\dots,R_k=0 \\ \sum_{i=1}^{k} R_i \le V}} (-V)_{\sum_{i=1}^{V} R_i} \frac{\prod_{i=1}^{k}\left(1+\alpha_i+\beta_i+V\right)_{R_i}\left(w_i\right)^{R_i}}{\prod_{i=1}^{k}\left\{\left(1+\alpha_i\right)_{R_i} R_i!\right\}}$$

$$\times \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\rho_1 R_1+\ldots+\rho_k R_k-s+pl)}{\Gamma(\mu+\rho_1 R_1+\ldots+\rho_k R_k-s+pl)} t^{\lambda+\rho_1 R_1+\ldots+\rho_k R_k+ml-s-1}$$

$$\times \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \left\{ \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) z^{-s} ds \right\} {}_{m+1}F_m \left\{ \begin{matrix} -n, I^* \\ J^* \end{matrix} \right\} D_x^n \{f(x)\} \quad (22)$$

$I^* = \lambda+\rho_1 R_1+\ldots+\rho_k R_k-s, \lambda+\rho_1 R_1+\ldots+\rho_k R_k-s+l,\ldots,$
$\lambda+\rho_1 R_1+\ldots+\rho_k R_k-s+(m-1)l$

$J^* = \mu+\rho_1 R_1+\ldots+\rho_k R_k-s, \mu+\rho_1 R_1+\ldots+\rho_k R_k-s+l,\ldots,$
$\mu+\rho_1 R_1+\ldots+\rho_k R_k-s+(m-1)l$

The conditions of validity of (22) can be easily get from the existence conditions given with theorem 1.

(iii) If in Theorem 2, the general class of polynomials is reduced to Multivariable Bessel Polynomial $y_{V,n_2,\ldots,n_k}^{\alpha_1,\ldots\alpha_k}$ represented in [14], then we get the following corollary:

Corollary 3

$$D_{l,\lambda-\mu,t}^{m} \left\{ t^{\lambda} y_{V,n_2,\ldots,n_k}^{\alpha_1\ldots\alpha_k}\left(2w_1 t^{\rho_1},\ldots,2w_k t^{\rho_k}\right) \aleph(zt) f(xt) \right\}$$

$$= \sum_{\substack{R_1,\ldots,R_k=0 \\ \sum_{i=1}^{k} R_i \le V}} (-V)_{\sum_{i=1}^{V} R_i} \frac{\prod_{i=1}^{k}(1+\alpha_1+V)_{R_1}}{R_1!} \prod_{i=2}^{k}(1+\alpha_i+n_i)_{R_i} \frac{(w_i)^{R_i}}{R_i!}$$

$$\times \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+\rho_1 R_1+\ldots+\rho_k R_k-s+pl)(1-\mu-\rho_1 R_1-\ldots-\rho_k R_k+s-pl)_n}{\Gamma(\mu+\rho_1 R_1+\ldots+\rho_k R_k-s+pl)(1-\lambda-\rho_1 R_1-\ldots-\rho_k R_k+s-pl)_n}$$

$$\times t^{\lambda+\rho_1 R_1+\ldots+\rho_k R_k+ml-s-1} \left\{ \frac{1}{2\pi i} \int_L \Omega_{P_i,Q_i,\tau_i;r}^{M,N}(s) z^{-s} ds \right\} {}_{m+1}F_m \left\{ \begin{matrix} -n, K^* \\ L^* \end{matrix} ; t \right\} D_x^n \{f(x)\} \quad (23)$$

$K^* = \lambda+\rho_1 R_1+\ldots+\rho_k R_k-s-n, \lambda+\rho_1 R_1+\ldots+\rho_k R_k-s-n+l,\ldots,$
$\lambda+\rho_1 R_1+\ldots+\rho_k R_k-s-n+(m-1)l$

$L^* = \mu+\rho_1 R_1+\ldots+\rho_k R_k-s-n, \mu+\rho_1 R_1+\ldots+\rho_k R_k-s-n+l,\ldots,$
$\mu+\rho_1 R_1+\ldots+\rho_k R_k-s-n+(m-1)l$

The conditions of validity of (23), can be easily obtained from the existence conditions given with theorem 2.

(iv) we put $\beta=\alpha$ and $\eta=0$ in equation (18), then we will get the following corollary:

Corollary 4

$$D_{0,x,\theta}^{\alpha,}\left[t^{w}\left(t^{\nu}+\xi^{\nu}\right)^{-\delta}S_{V}^{U_1,\dots,U_k}\left\{Y_1 t^{\rho_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_1},\dots,Y_k t^{\rho_k}\left(t^{\nu}+\xi^{\nu}\right)^{-\sigma_k}\right\}\right.$$

$$\left.\times\aleph\left(z_1 t^{\lambda_1}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_1},\dots,z_p t^{\lambda_p}\left(t^{\nu}+\xi^{\nu}\right)^{-\mu_p}\right)\right](x)$$

$$=x^{w-\theta\alpha+\sum_{i=1}^{k}\rho_i R_i}\,\xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_i R_i}\sum_{R_1,\dots,R_k=0}^{\sum_{i=1}^{k}U_i R_i\le V}(-V)_{\sum_{i=1}^{V}U_i R_i}$$

$$\times A\left(V,R_1,\dots,R_k\right)\frac{Y_1^{R_1}\dots Y_k^{R_k}}{R_1!\dots R_k!}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-x}{\xi}\right)^{m}$$

$$\times\aleph_{P_i+2,Q_i+2,\tau_i,R;U_{11}}^{0,N+2;m_1,n_1;\dots;m_p,n_p}\left[\begin{array}{c|c} z_1\xi^{-\mu_1\nu}x^{\lambda_1} & K^{*}\\ \vdots & \\ z_p\xi^{-\mu_p\nu}x^{\lambda_p} & L^{*}\end{array}\right] \tag{24}$$

Where

$$U_{11}=p_i(1),q_i(1),\tau_i(1);R^{(1)};\dots;p_i(r),q_i(r),\tau_i(r);R^{(r)}$$

$$K^{*}=\left(a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,n}{}'\left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\dots\mu_p\right),\left(\frac{-w-m\nu-\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\dots,\frac{\lambda_p}{\theta}\right),$$

$$,\left[\tau_i\left(a_{ji},\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)\right]_{n+1,P_i};\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i(1)}\left(c_{ji(1)}^{(1)},\gamma_{ji(1)}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}}$$

$$;\dots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i(r)}\left(c_{ji(r)}^{(r)},\gamma_{ji(r)}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

$$L^* = \left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i, \mu_1, \ldots \mu_p\right), \left(\alpha - \frac{w+m\nu+\sum_{i=1}^{k}\rho_i R_i}{\theta}, \frac{\lambda_1}{\theta}, \ldots, \frac{\lambda_p}{\theta}\right), \left[\tau_i\left(b_{ji}, \beta_j^{(1)}, \ldots, \beta_j^{(r)}\right)\right]_{m+1,Q_i}$$

$$;\left[\left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,n_1}\right], \left[\tau_{i(1)}\left(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}\right)\right]_{m_1+1,Q_i^{(1)}}$$

$$;\ldots;\left[\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}\right], \left[\tau_{i(r)}\left(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}\right)\right]_{m_r+1,Q_i^{(r)}}$$

The conditions of validity of (24) can be easily obtained from the existence conditions given with theorem 3.

(v) If we put $\theta = 1$ in Theorem 3, we get the following corollary:

Corollary 5

Let $0 \le \alpha < 1, \beta, \eta, x \in \Re, \theta \in N, \mathrm{Re}(\alpha) > 0, w, \delta > 0, \rho_j, \sigma_j > 0 \ (j = 1, \ldots, k), \lambda, \mu > 0, z \in C$. If the existence conditions of Aleph ($\aleph$) function are satisfied, then the generalized fractional derivative $D_{0,x,\theta}^{\alpha,\beta,\eta}$ of the product of Aleph ($\aleph$) function and $S_V^{U_1,\ldots,U_k}$ exists and there holds the following formula:

$$D_{0,x,1}^{\alpha,\beta,\eta}\left[t^w\left(t^\nu+\xi^\nu\right)^{-\delta} S_V^{U_1,\ldots,U_k}\left\{Y_1 t^{\rho_1}\left(t^\nu+\xi^\nu\right)^{-\sigma_1}, \ldots, Y_k t^{\rho_k}\left(t^\nu+\xi^\nu\right)^{-\sigma_k}\right\}\right.$$

$$\left.\times \aleph\left(z_1 t^{\lambda_1}\left(t^\nu+\xi^\nu\right)^{-\mu_1}, \ldots, z_p t^{\lambda_p}\left(t^\nu+\xi^\nu\right)^{-\mu_p}\right)\right](x)$$

$$= x^{w-\beta+\sum_{i=1}^{k}\rho_i R_i} \xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_i R_i} \sum_{\substack{R_1,\ldots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{V} U_i R_i}$$

$$\times A\left(V, R_1, \ldots, R_k\right) \frac{Y_1^{R_1} \ldots Y_k^{R_k}}{R_1! \ldots R_k!} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-x}{\xi}\right)^m$$

$$\times \aleph_{P_i+3, Q_i+3, \tau_i, R; U_{11}}^{0,N+3; m_1,n_1; \ldots; m_p,n_p}\left[\begin{array}{c|c} z_1 \xi^{-\mu_1 \nu} x^{\lambda_1} & M^* \\ \vdots & \\ z_p \xi^{-\mu_p \nu} x^{\lambda_p} & N^* \end{array}\right] \quad (25)$$

Where

$$U_{11}=p_i(1),q_i(1),\tau_i(1);R^{(1)};\ldots;p_i(r),q_i(r),\tau_i(r);R^{(r)}$$

$$M^*=\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}{}',\left(1-\delta-m-\sum_{i=1}^{k}\sigma_iR_i,\mu_1,\ldots\mu_p\right),\left(-w-m\nu-\sum_{i=1}^{k}\rho_iR_i,\lambda_1,\ldots,\lambda_p\right),$$

$$\left(\beta-\eta-\left(w+m\nu+\sum_{i=1}^{k}\rho_iR_i\right),\lambda_1,\ldots,\lambda_p\right),\left[\tau_i\left(a_{ji},\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)\right]_{n+1,P_i};\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,n_1}\right],$$

$$\left[\tau_{i(1)}\left(c_{ji(1)}^{(1)},\gamma_{ji(1)}^{(1)}\right)\right]_{n_1+1,P_i^{(1)}};\ldots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i(r)}\left(c_{ji(r)}^{(r)},\gamma_{ji(r)}^{(r)}\right)\right]_{n_r+1,P_i^{(r)}}$$

$$N^*=\left(1-\delta-\sum_{i=1}^{k}\sigma_iR_i,\mu_1,\ldots\mu_p\right),\left(\beta-\left(w+m\nu+\sum_{i=1}^{k}\rho_iR_i\right),\lambda_1,\ldots\lambda_p\right),\left(\alpha-\eta-\left(w+m\nu+\sum_{i=1}^{k}\rho_iR_i\right),\lambda_1,\ldots\lambda_p\right),$$

$$\left[\tau_i\left(b_{ji},\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)\right]_{m+1,q_i};\left[\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i(1)}\left(d_{ji(1)}^{(1)},\delta_{ji(1)}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}}$$

$$;\ldots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i(r)}\left(d_{ji(r)}^{(r)},\delta_{ji(r)}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}}$$

Corollary 6

On taking $\beta=\alpha$ and $\eta=0$, the above result can be further reduced to following result for the Riemann-Liouville Fractional derivative operator:

$$D_{0,x,1}^{\alpha,}\left[t^w\left(t^\nu+\xi^\nu\right)^{-\delta}S_V^{U_1,\ldots,U_k}\left\{Y_1t^{\rho_1}\left(t^\nu+\xi^\nu\right)^{-\sigma_1},\ldots,Y_kt^{\rho_k}\left(t^\nu+\xi^\nu\right)^{-\sigma_k}\right\}\right.$$

$$\left.\times\aleph\left(z_1t^{\lambda_1}\left(t^\nu+\xi^\nu\right)^{-\mu_1},\ldots,z_pt^{\lambda_p}\left(t^\nu+\xi^\nu\right)^{-\mu_p}\right)\right](x)$$

$$=x^{w-\alpha+\sum_{i=1}^{k}\rho_iR_i}\,\xi^{-\nu\delta-\nu\sum_{i=1}^{k}\sigma_iR_i}\sum_{\substack{R_1,\ldots,R_k=0\\ \sum_{i=1}^{k}U_iR_i\le V}}(-V)_{\sum_{i=1}^{V}U_iR_i}$$

$$\times A(V,R_1,\ldots,R_k)\frac{Y_1^{R_1}\ldots Y_k^{R_k}}{R_1!\ldots R_k!}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-x}{\xi}\right)^m$$

$$\times\aleph_{P_i+2,Q_i+2,\tau_i,R;U_{11}}^{0,N+2;m_1,n_1;\ldots;m_p,n_p}\left[\begin{array}{c|c}z_1\xi^{-\mu_1\nu}x^{\lambda_1} & O^*\\ \vdots & \\ zp\xi^{-\mu_p\nu}x^{\lambda_p} & P^*\end{array}\right]\tag{26}$$

Where

$$U_{11} = P_i(1), Q_i(1), \tau_i(1); R^{(1)}; \ldots; P_i(r), Q_i(r), \tau_i(r); R^{(r)}$$

$$O^* = \left(a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)_{1,n}, \left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i, \mu_1, \ldots \mu_p\right), \left(-w-m\nu-\sum_{i=1}^{k}\rho_i R_i, \lambda_1, \ldots, \lambda_p\right),$$

$$\left[\tau_i\left(a_{ji}, \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)\right]_{n+1, P_i}; \left[\left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,n_1}\right],$$

$$\left[\tau_{i(1)}\left(c_{ji(1)}^{(1)}, \gamma_{ji(1)}^{(1)}\right)\right]_{n_1+1, P_i^{(1)}}; \ldots; \left[\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,n_r}\right], \left[\tau_{i(r)}\left(c_{ji(r)}^{(r)}, \gamma_{ji(r)}^{(r)}\right)\right]_{n_r+1, P_i^{(r)}}$$

$$P^* = \left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i, \mu_1, \ldots \mu_p\right), \left(\alpha - \left(w+m\nu+\sum_{i=1}^{k}\rho_i R_i\right), \lambda_1, \ldots \lambda_p\right),$$

$$\left[\tau_i\left(b_{ji}, \beta_j^{(1)}, \ldots, \beta_j^{(r)}\right)\right]_{m+1, Q_i}; \left[\left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,n_1}\right], \left[\tau_{i(1)}\left(d_{ji(1)}^{(1)}, \delta_{ji(1)}^{(1)}\right)\right]_{m_1+1, Q_i^{(1)}}$$

$$; \ldots; \left[\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,m_r}\right], \left[\tau_{i(r)}\left(d_{ji(r)}^{(r)}, \delta_{ji(r)}^{(r)}\right)\right]_{m_r+1, Q_i^{(r)}}$$

(vi) In theorem (3), $\tau_i = 1$ and $R = 1, \nu = 1$, then Aleph ($\aleph$) function reduce to H function [11]

$$D_{0,x,\theta}^{\alpha,\beta,\eta}\left[t^w (t+\xi)^{-\delta} S_V^{U_1,\ldots,U_k}\left\{Y_1 t^{\rho_1}(t+\xi)^{-\sigma_1}, \ldots, Y_k t^{\rho_k}(t+\xi)^{-\sigma_k}\right\}\right.$$

$$\left.\times H\left(z_1 t^{\lambda_1}(t+\xi)^{-\mu_1}, \ldots, z_p t^{\lambda_p}(t+\xi)^{-\mu_p}\right)\right](x)$$

$$= x^{w-\theta\beta+\sum_{i=1}^{k}\rho_i R_i} \xi^{-\delta-\sum_{i=1}^{k}\sigma_i R_i} \sum_{\substack{R_1,\ldots,R_k=0 \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i}$$

$$\times A(V, R_1, \ldots, R_k) \frac{Y_1^{R_1}, \ldots, Y_k^{R_k}}{R_1!, \ldots, R_k!} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-x}{\xi}\right)^m$$

$$\times H_{P+3, Q+3, P_1, Q_1; \ldots; P_p, Q_p}^{0, N+3; m_1, n_1; \ldots; m_p, n_p}\left[\begin{array}{c|c} z_1 \xi^{-\mu_1} x^{\lambda_1} & R^* \\ \vdots & \\ z_p \xi^{-\mu_p} x^{\lambda_p} & S^* \end{array}\right] \tag{27}$$

Where

$$R^{*}=\left(1-\delta-m-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\ldots\mu_p\right),\left(\frac{-w-m-\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left(\beta-\eta-\frac{w+m+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,P};\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P_r}$$

$$S^{*}=\left(1-\delta-\sum_{i=1}^{k}\sigma_i R_i,\mu_1,\ldots\mu_p\right),\left(\beta-\frac{w+m+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),\left(\alpha-\eta-\frac{w+m+\sum_{i=1}^{k}\rho_i R_i}{\theta},\frac{\lambda_1}{\theta},\ldots,\frac{\lambda_p}{\theta}\right),$$

$$\left(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,Q};\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q_r}$$

(vii) If we put k=1 and $\aleph(z)=1$ in theorem 1 and 2, then we find at a known results due to Goyal and Saxena [2].

(viii) If we reduce the generalized class of polynomial to unity and $\aleph(z)=1$ in theorem 1 and 2, then we arrive at known results due to Dube [5].

(ix) Many other special case of H function is reducing to other function in [3].

APPLICATIONS OF THE THEOREM 1 AND 2

Now we establish two multiplication formule as application of Theorem 1 and 2 respectively.

Let $f(x)=x^{\sigma-1}\aleph\left[xy_1,\ldots,xy_n\right]$ (28)

Now using (11) on $f(x)$, then we get

(i) $$D_x^n\left\{x^{\sigma-1}\aleph\begin{bmatrix}xy_1\\ \vdots\\ xy_r\end{bmatrix}\right\}=x^{\sigma-n-1}\aleph^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P_i+1,Q_i+1;\tau_i,R;U_{11}}\left[\begin{matrix}xy_1\\ \vdots\\ xy_n\end{matrix}\middle|\begin{matrix}(1-\sigma;1,1),*\\ \\ (1-\sigma+n;1,1),*\end{matrix}\right] \tag{29}$$

Also in view of (1), (2) and (8), we have

(ii) $$D^m_{l,\lambda-\mu,t}\left\{t^{\lambda+\sigma-2}x^{\sigma-1}S_V^{U_1,\ldots,U_k}\left\{w_1t^{\rho_1},\ldots,w_kt^{\rho_k}\right\}\aleph\left(xt,xy_1t,\ldots,xy_{n-1}t\right)\right\}$$

$$=t^{\lambda+\sigma+\rho_1R_1+\ldots+\rho_kR_k+ml-2}\sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k}U_iR_i\le V}(-V)_{\sum_{i=1}^{k}U_iR_i}$$

$$\times A(V,R_1,\ldots,R_k)\frac{w_1^{R_1}\ldots w_k^{R_k}}{R_1!\ldots R_k!}x^{\sigma-1}$$

$$\times \aleph^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P_i+m,Q_i+m;\tau_i,R,U_{11}}\left[\begin{array}{c|c} x y_1 t & T^* \\ x y_n t & U^* \end{array}\right] \tag{30}$$

$T^* = \left(2-\lambda-\sigma-\rho_1 R_1-\ldots-\rho_k R_k, 1,\ldots,1\right), \left(2-\lambda-\sigma-\rho_1 R_1-\ldots-\rho_k R_k-l, 1,\ldots,1\right),\ldots,$

$\left(2-\lambda-\sigma-\rho_1 R_1-\ldots-\rho_k R_k-(m-1)l, 1,\ldots,1\right), **$

$U^* = \left(2-\mu-\sigma-\rho_1 R_1-\ldots-\rho_k R_k, 1,\ldots,1\right), \left(2-\mu-\sigma-\rho_1 R_1-\ldots-\rho_k R_k-l, 1,\ldots,1\right),\ldots,$

$\left(2-\mu-\sigma-\rho_1 R_1-\ldots-\rho_k R_k-(m-1)l, 1,\ldots,1\right), **$

Where the asterisk $**$ in (29) and (30) indicates that the parameters at these places are the same as the parameters of the multivariable's Aleph-function (8).

Substituting the values of differential operators (29) and (30) in theorem 1 and 2, $y_1 = \frac{y_1}{x}, \quad , y_n = \frac{y_n}{x}$

Comparing the coefficients of $w_i^{R_i}$ both sides and replacing $\lambda \to \lambda - \rho_1 R_1 - \ldots - \rho_k R_k$, $\mu \to \mu - \rho_1 R_1 - \ldots - \rho_k R_k$ and

$\sigma \to 1-\sigma$, we get the following Multiplication formula for multivariable's Aleph-function.

$(1-\lambda+\sigma, 1,\ldots,1), (1-\lambda+\sigma-l, 1,\ldots,1),\ldots, (1-\lambda+\sigma-(m-1)l, 1,\ldots,1), **$

Multiplication Formula 1

$$\aleph^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P_i+m,Q_i+m;\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 t & X^* \\ y_n t & Y^* \end{array}\right]$$

$$= t^{\sigma} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{p=0}^{m-1} \frac{\Gamma(\lambda+pl)}{\Gamma(\mu+pl)} \, {}_{m+1}F_m\left\{\begin{array}{c} -n, \lambda, \lambda+l, \ldots, \lambda+(m-1)l \\ \mu, \mu+l, \ldots, \mu+(m-1)l \end{array}; t\right\}$$

$$\times \aleph^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P_i+1,Q_i+1,\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 & (\sigma,1,\ldots,1), ** \\ y_n & (\sigma+n,1,\ldots,1), ** \end{array}\right] \tag{31}$$

$X^* = (1-\lambda+\sigma, 1,\ldots,1), (1-\lambda+\sigma-l, 1,\ldots,1),\ldots,(1-\lambda+\sigma-(m-1)l, 1,\ldots,1), *$

$Y^* = (1-\mu+\sigma, 1,\ldots,1), (1-\mu-\sigma-l, 1,\ldots,1),\ldots,(1-\mu-\sigma-(m-1)l, 1,\ldots,1), *$

Multiplication Formula 2

$$\aleph^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P_i+m,Q_i+m;\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 t & W^* \\ y_n t & Z^* \end{array}\right]$$

$$= t^{\sigma-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pl)(1-\mu-pl)_n}{\Gamma(\mu+pl)(1-\lambda-pl)_n}\right]{}_{m+1}F_m\left\{\begin{array}{c}-n,\lambda\text{-n},\lambda-n+l,\ldots,\lambda\text{-n}+(m-1)l\\ \mu\text{-n},\mu\text{-n}+l,\ldots,\mu\text{-n}+(m-1)l\end{array};t\right\}$$

$$\times\aleph^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P_i+1,Q_i+1,\tau_i,R,U_{11}}\left[\begin{array}{c|c} y_1 & (\sigma+n,1,\ldots,1),** \\ y_n & (\sigma,1,\ldots,1),** \end{array}\right]$$

$$W^*=(-\lambda+\sigma,1,\ldots,1),(-\lambda+\sigma-l,1,\ldots,1),\ldots,(-\lambda+\sigma-(m-1)l,1,\ldots,1),*$$

$$Z^*=(-\mu+\sigma,1,\ldots,1),(-\mu-\sigma-l,1,\ldots,1),\ldots,(1-\mu-\sigma-(m-1)l,1,\ldots,1),*$$

(32)

4. Special Cases for Multiplication Formulae

$\tau_i = 1$ and $R = 1$,in equation (31) and (32) respectively then Aleph ($\aleph$) function reduce to H function [3]

(i) **Multiplication Formula 1.**

$$H^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P+m,Q+m;P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{array}{c|c} y_1 t & X^* \\ y_n t & Y^* \end{array}\right]$$

$$= t^{\sigma}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\prod_{p=0}^{m-1}\frac{\Gamma(\lambda+pl)}{\Gamma(\mu+pl)}\,{}_{m+1}F_m\left\{\begin{array}{c}-n,\lambda,\lambda+l,\ldots,\lambda+(m-1)l\\ \mu,\mu+l,\ldots,\mu+(m-1)l\end{array};t\right\}$$

$$\times H^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P+1,Q+1,P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{array}{c|c} y_1 & (\sigma,1,\ldots,1),** \\ y_n & (\sigma+n,1,\ldots,1),** \end{array}\right]$$

$$X^*=(1-\lambda+\sigma,1,\ldots,1),(1-\lambda+\sigma-l,1,\ldots,1),\ldots,(1-\lambda+\sigma-(m-1)l,1,\ldots,1),*$$

$$Y^*=(1-\mu+\sigma,1,\ldots,1),(1-\mu-\sigma-l,1,\ldots,1),\ldots,(1-\mu-\sigma-(m-1)l,1,\ldots,1),*$$

(33)

(ii) **Multiplication Formula 2.**

$$H^{0,N+m,m_1,n_1;\ldots;m_r,n_r}_{P+m,Q+m;P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{array}{c|c} y_1 t & W^* \\ y_n t & Z^* \end{array}\right]$$

$$= t^{\sigma-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\prod_{p=0}^{m-1}\left[\frac{\Gamma(\lambda+pl)(1-\mu-pl)_n}{\Gamma(\mu+pl)(1-\lambda-pl)_n}\right]{}_{m+1}F_m\left\{\begin{matrix}-n,\lambda\text{-}n,\lambda-n+l,\ldots,\lambda\text{-}n+(m-1)l\\ \mu\text{-}n,\mu\text{-}n+l,\ldots,\mu\text{-}n+(m-1)l\end{matrix};t\right\}$$

$$\times H^{0,N+1,m_1,n_1;\ldots;m_r,n_r}_{P+1,Q+1,P_1,Q_1;\ldots;P_p,Q_p}\left[\begin{matrix}y_1\\ \vdots\\ y_n\end{matrix}\middle|\begin{matrix}(\sigma+n,1,\ldots,1),**\\ (\sigma,1,\ldots,1),**\end{matrix}\right]$$

$$W^*=(-\lambda+\sigma,1,\ldots,1),(-\lambda+\sigma-l,1,\ldots,1),\ldots,(-\lambda+\sigma-(m-1)l,1,\ldots,1),*$$

$$Z^*=(-\mu+\sigma,1,\ldots,1),(-\mu-\sigma-l,1,\ldots,1),\ldots,(1-\mu-\sigma-(m-1)l,1,\ldots,1),* \tag{34}$$

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