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# Homotopy Analysis Method to Solve Fuzzy Impulsive Fractional Differential Equations

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**Abstract:** In this paper, we study semi-analytical methods entitled Homotopy Analysis Method (HAM) to solve fuzzy impulsive fractional differential equations based on the concept of generalized Hukuhara differentiability. At the end first of Homotopy Analysis Method is defined and its properties are considered completely. Then econvergence theorem for the solution are proved and we will show that the approximate solution convergent to the exact solution. Numerical example indicate that this method can be easily applied to many linear and nonlinear problems.

**Keywords:** Homotopy Analysis Method, Fuzzy impulsive fractional differential, generalized Hukuhara differentiability.

## 1. Introduction

Fractional calculus is a new powerful tool which has been recently employed to model complex biological systems with non-linear behavior and long-term memory. The fractional derivatives have several different kinds of definitions, among which the RiemannLiouville fractional derivative and the Caputo fractional derivative are two of the most important ones in applications. Fractional (fractional-order) derivative is a generalization of integer-order derivative and integral. It originated in the letter about the meaning of 1/2 order derivative from LHopital to Leibnitz in 1695, and is a promising tool for describing memory phenomena. The kernel function of fractional derivative is called memory function, but it does not reflect any physical process. One of the important branches of this theory is impulsive fractional differential equations. The idea of the theory of impulsive fractional differential equation has been emerging as an effective tool area of investigation in recent years (see [13, 14, 17, 31, 32]). JinRong. Wang, W. Wei and YanLong. Yang [43] solving impulsive fractional differential equations in banach spaces. Qi Wang, Dicheng Lu, Yayun Fang [44] show that the impulsive fractional differential is stable. Also fuzzy set theory is the significant tool for modeling unknown problems and can be found in many branches of regional, physical, mathematical and engineering sciences. The concept of the fuzzy set theory was first proposed by Zadeh, Zimmerman and Kaleva (see[20, 46, 47]). As a result many things can happen in the real world has a fuzzy meaning. One of the very important branches of the fuzzy theory is fuzzy impulsive fractional differential equations. As an advantage of the fuzzy impulsive fraction differential equations is, they are able to describe the solution of the model at the certain moments with more sharpness and rapidly changes in their states. While the classical differential equations are not able to describe the mentioned behavior. Fuzzy impulsive fractional differential equations are usually hard to solve analytically and the exact solution is rather difficult to be obtained. But the idea of fuzzy impulsive fractional differential equations has been studied by scientists and engineers such as N.Najafi and T. Allaviranloo [23, 24, 26]. They have considered new method to solve fuzzy impulsive fractional differential equation based on fuzzy fractional differential transform method as one of the branches of fuzzy impulsive fractional differential equations. Our aim in this paper is to study the semi-analytical methods for solving fuzzy impulsive fractional differential equations. We will use the Homotopy Analysis Method (HAM) based on generalized Hukuhara differentiability to solve a nonlinear and linear fuzzy impulsive fractional differential equations given by

$$ {}_cD_*^{\frac{1}{\alpha}} y(t) = f(t,y), \quad t \in J = [0,T], \quad t \neq t_k, \tag{1.1} $$

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad t = t_k, \quad k = 0,1,\ldots,m, \tag{1.2} $$

$$ y(0) = y_0, \tag{1.3} $$

In this paper the set of all fuzzy numbers is denoted $\mathbb{R}_f$. where $m-1 < \frac{1}{a} \leq m$ , ${}_cD_*^{\frac{1}{a}}$ denotes the Caputo fractional generalized derivative of order $\frac{1}{a}$ , and $f : J \times \mathbb{R}_f \to \mathbb{R}_f$ , is continuous fuzzy function, $I_k : \mathbb{R}_f \to \mathbb{R}_f$ , $y_0 \in \mathbb{R}_f$ , $0 = t_0 < t_1 < \cdot \cdot \cdot < t_m < t_{m+1} = T$, $\Delta|_{t=t_k} = y(t_k^+) \ominus y(t_k^-)$, $y(t_k^+)=\log_{h\to 0^+} y(t_k+h)$, and $y(t_k^-) = \log_{h\to 0^-} y(t_k+h)$ represent the right and left limits of y(t) at t = $t_k$. The paper is organized as follow: We describe the basic notation and prelimition in Section 2. We describe the fuzzy fractional impulsive differential equation in Section 3. The HAM is presented in Section 4. In Section 5, we solving fuzzy impulsive fractional differential equation by homotopy analysis method based on the concept of generalized Hukuhara differentiability. Numerical example is given to clarify the details and efficiency of the method in Section 6. This paper ends with conclusion in Section 7.

## 2. Preliminaries

In this section, we give some basic definitions, theorems and properties of fractional calculus theory which are further used in this paper.

**Definition 2.1.** ([22]) We represent an arbitrary fuzzy number by an ordered pair function $(\underline{u}(r), \overline{u}(r))$, which satisfies the following requirements:

a: $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

b: $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

c: $\underline{u}(r) \leq \overline{u}(r)$ , $0 \leq r \leq 1$

**Definition 2.2.** ([27]) A crisp number $\theta$ is simply represented by $\underline{u}(t,r) = \overline{u}(t,r) = \theta$, $0 \leq r \leq 1$. We recall that for $a < b < c$ which $a, b, c \in \mathbb{R}$, the triangular fuzzy number $u = (a, b, c)$ is determined by $a, b, c$ such that $\underline{u}(t,r) = a + (b-c)r$ and $\overline{u}(t,r) = c - (c-b)r$ are left branch and right branch, for all $r \in [0,1]$.

**Definition 2.3.** [24] Let $u, v \in \mathbb{R}_F$ . If there exists $w \in \mathbb{R}_F$ such that $u = v \oplus w$, then $w$ is called the H-difference of $u$ and $v$, and it is denoted by $w = u \ominus_H v$. The Hukuhara difference is also motivated by the problem of inverting the addition, if $x, y$ are crisp numbers then $(x+y) - y = x$ but this is not true if $x, y$ are fuzzy numbers. If $u$ and $v$ are fuzzy numbers (and not in general fuzzy sets), then $(u+v) \ominus_H v = u$ i.e. the H-difference inverts the addition of fuzzy numbers. The gH-difference for fuzzy numbers can be defned as follows:

**Definition 2.4.** ([28]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathbb{R}_F$ is defined as follows:

$$u \ominus_{gH} v = w \iff \begin{cases} (i) & u = v \oplus w, \\ or & \\ (ii) & v = u \oplus (-1)w. \end{cases}$$

It is easy to show that $(i)$ and $(ii)$ are both valid if and only if $w$ is a crisp number. In terms of $r$- cuts we have $[u \ominus_{gH} v]_r = [\min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}, \max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}]$ and if the H-difference exists, then $u \ominus_H v = u \ominus_{gH} v$, the conditions for the existence of $u \ominus_{gH} v = w \in \mathbb{R}_F$ are

$$Case(i) = \begin{cases} \underline{w}_r = \underline{u}_r - \underline{v}_r \quad and, \quad \overline{w}_r = \overline{u}_r - \overline{v}_r \\ with \quad \underline{w}_r \ increasing, \quad \overline{w}_r \ decreasing \quad and \quad \underline{w}_r \leq \overline{w}_r \end{cases}$$

$$Case(ii) = \begin{cases} \underline{w}_r = \overline{u}_r - \overline{v}_r \quad and, \quad \overline{w}_r = \underline{u}_r - \underline{v}_r \\ with \quad \underline{w}_r \ increasing, \quad \overline{w}_r \ decreasing \quad and \quad \underline{w}_r \leq \overline{w}_r \end{cases}$$

The condition for the existence of $u \ominus_{gH} v \in \mathbb{R}_F$ are given in ([22, 24, 30]. Please note that a function $f : [a,b] \to \mathbb{R}_F$ so called fuzzy number valued function. The $r$-level representation of fuzzy-valued function $f$ is expressed by $[f(t)]_r = [\underline{f}(t,r), \overline{f}(t,r)], t \in [a,b], r \in [0,1]$.

**Definition 2.5.** ([12]) For $0 < r \leq 1$ denote $[u]_r = \{t \in \mathbb{R} | u(t) \geq r\} = [\underline{u}(r), \overline{u}(r)]$ and for $r = 0$ by the closure of the support $[u]_0 = cl\{t | t \in \mathbb{R}, u(t) > 0\}$ where $cl$ denotes the closure of a subset. The addition $u \oplus v$ and the scaler multiplication $k \odot u$ are defined as having the level cuts

$$[u \oplus v]_r = [u]_r \oplus [v]_r = \{x + y | x \in [u]_r, y \in [v]_r\}, \tag{2.1}$$

and

$$[k \odot u]_r = k.[u]_r = \begin{cases} [k\underline{u}(r), k\overline{u}(r)], 0 \leq k, \\ [k\overline{u}(r), k\underline{u}(r)], k < 0. \end{cases}$$

The subtraction of fuzzy numbers $u \ominus_{gH} v$ is defined as the addition $u \oplus (-1)v$, if $v(r) = [\underline{v}(r), \overline{v}(r)]$ where $[(-1) \odot v]_r = [-\overline{v}(r), -\underline{v}(r)]$.

**Definition 2.6.** The Hausdorff distances between fuzzy numbers is given by $d : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ as in [4].

$$d(u, v) = \sup_{0<r\leq 1} \max \Big( |\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)| \Big).$$

Consider $u, v, w, z \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, then the following properties are well-known for metric $d$

1) $d(u \oplus w, v \oplus w) = d(u, v)$,

2) $d(\lambda u, \lambda v) = |\lambda| d(u, v)$,

3) $d(u \oplus v, w \oplus z) \leq d(u, w) + d(v, z)$,

4) $d(u \ominus_H v, w \ominus_H z) \leq d(u, w) + d(v, z)$,

as long as $u \ominus_H v$ and $w \ominus_H z$ exist, where $u, v, w, z \in \mathbb{R}_F$.

**Theorem 2.1.** *( [45]) Let $f(t)$ be a fuzzy- valued function on $[a, \infty)$ and it is represented by $(\underline{f}(t, r), \ \overline{f}(t, r))$. For any fixed $r \in [0, 1]$, assume $\overline{f}(t, r)$and $\underline{f}(t, r)$ are Riemann-integrable on $[a, b]$ for every $b \geq a$ and assume there are two positive values $\underline{M}(r)$ and $\overline{M}(r)$ such that*

$$\int_a^b |\underline{f}(t, r)| dx \leq \underline{M}(r),$$

*and*

$$\int_a^b |\overline{f}(t, r)| dx \leq \overline{M}(r),$$

*for every $b \geq a$. Then $f(t)$ is improper fuzzy Riemann-integrable on $[a, \infty)$ and is a fuzzy number. Further more, we have:*

$$[\int_a^\infty f(t) dt]_r = [\int_a^\infty \underline{f}(t, r) dt, \int_a^\infty \overline{f}(t, r) dt].$$

**Definition 2.7.** [7] let $f : [a, b] \longrightarrow \mathbb{R}_F$ is called fuzzy continuous if for arbitrary fixed $t_0 \in \mathbb{R}_F$ and $\xi > 0$, there exists an $\delta > 0$, such that if

$$|t - t_0| < \delta, then \ \ d(f(t), f(t_0)) < \xi$$

**Lemma 1.** $\forall \ \alpha > 0$ *and* $\gamma > -1$

$$\int_0^t (t - s)^{k\alpha - 1} s^\gamma ds = \frac{\Gamma(k\alpha)\Gamma(\gamma + 1)}{\Gamma(k\alpha + \gamma + 1)} t^{k\alpha + \gamma}.$$

*Where $\Gamma$ is the gamma function and defined by*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

***Proof:*** *Lemma 2.6 [43].*

**Definition 2.8.** The generalized Hukuhara derivative of a fuzzy number valued function $f : (a, b) \to \mathbb{R}_F$ at $t_0$ is defined

$$f'_{gH}(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h} \tag{2.2}$$

If $f'_{gH}(t_0) \in \mathbb{R}_F$ satisfying (2.5) exists, we say that $f$ is generalized Hukuhara differentiable (gH-differentiable for short) at $t_0$. Also we say that $f$ is [(i)-gH]-diferentiable at $t_0$ if

$$(i) \ \ [f'_{i.gH}(t_0)]_r = [\underline{f}'(t_0, r), \ \overline{f}'(t_0, r)] \tag{2.3}$$

and if $f$ is [(ii)-gH]-diferentiable at $t_0$ if

$$(ii) \ \ [f'_{ii.gH}(t_0)]_r = [\overline{f}'(t_0, r), \underline{f}'(t_0, r)] \tag{2.4}$$

**Definition 2.9.** ([4]) Let $f : (a, b) \to \mathbb{R}_F$. We say that $f$ is gH- differentiable of the $m^{th}$order at $t_0$ whenever the function $f$ is gH-differentiable of the order $j, j = 1, 2, \ldots, n-1$ , at $t_0$ provided that gH-differentiable type has no change, then there exist $f_{gH}^{(m)}(t_0) \in \mathbb{R}_F$ such that

$$f_{gH}^{(m)}(t_0) = \lim_{h \to 0} \frac{f_{gH}^{(m-1)}(t_0 + h) \ominus_{gH} f_{gH}^{(m-1)}(t_0)}{h}, m \in \mathbb{N} \tag{2.5}$$

**Definition 2.10.** ([22]) Let $f : [a, b] \to \mathbb{R}_F$, the fuzzy fractional derivative of $f(t)$ in the Caputo sense is defned as

$${}_cD_*^{\frac{1}{\alpha}} f(t) = \frac{1}{\Gamma(m - \frac{1}{\alpha})} \int_a^t \frac{f^{(m)}(t)\mathrm{d}s}{(t-s)^{\frac{1}{\alpha} - m + 1}}, \quad m - 1 < \alpha < m, t > a \tag{2.6}$$

**Definition 2.11.** ([10]) Let $f : [a, b] \to \mathbb{R}_F$, the fuzzy Riemann-Liouville integral of fuzzy number valued function $f$ is defined as follows:

$$(I_{a|t}^{\frac{1}{\alpha}} f)(t) = \frac{1}{\Gamma(\frac{1}{\alpha})} \int_a^t \frac{f(t)}{(t-s)^{1-\frac{1}{\alpha}}} \mathrm{d}s, \ t > a,$$

where

$$[I_{a|t}^{\alpha} f(t)]_r = \left[\frac{1}{\Gamma(\frac{1}{\alpha})} \int_a^t \frac{\underline{f}(t,r)}{(t-s)^{1-\frac{1}{\alpha}}} \mathrm{d}s, \ \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\overline{f}(t,r)}{(t-s)^{1-\frac{1}{\alpha}}} \mathrm{d}s\right],$$

for $a \le s \le t$ and $0 < \alpha \le 1$.

**Definition 2.12.** ([22])The Caputo generalized Hukuhara differentiability of fuzzy number valued function $f([gH]$-differentiability for short), where $t > a$, is defned as following:

$${}_cD_*^{\frac{1}{\alpha}} f_{gH}(t) = I_{a|t}^{m-\frac{1}{\alpha}} f_{gH}^{(m)}(t) = \frac{1}{\Gamma(m - \frac{1}{\alpha})} \int_a^t \frac{f_{gH}^{(m)}(t)\mathrm{d}s}{(t-s)^{\frac{1}{\alpha} - m + 1}}, \quad m - 1 < \frac{1}{\alpha} < m, t > a \tag{2.7}$$

We suppose that any order of differentiability of fuzzy function $f$ exist in the sense of gH. Moreover we say that $f$ is $[(i) - gH]$-Caputo differentiable at $t$ if

$$[{}_cD_{i.gH}^{\frac{1}{\alpha}} f(t)]_r = [{}_cD_*^{\frac{1}{\alpha}} \underline{f}(t,r), \ {}_cD_*^{\frac{1}{\alpha}} \overline{f}(t,r)], \tag{2.8}$$

as well as $f$ is $[(ii) - gH]$-Caputo differentiable at $t$ if

$$[{}_cD_{ii.gH}^{\frac{1}{\alpha}} f(t)]_r = [{}_cD_*^{\frac{1}{\alpha}} \overline{f}(t,r), \ {}_cD_*^{\frac{1}{\alpha}} \underline{f}(t,r)], \tag{2.9}$$

where $m - 1 < \frac{1}{\alpha} < m,_c D_*^{\frac{1}{\alpha}} \overline{f}(t,r)$ and $cD_*^{\frac{1}{\alpha}} \underline{f}(t,r)$ are defned in Defnition 2.10.

**Definition 2.13.** ([12]) We say that a point $t_0 \in (a, b)$, is a switching point for the differentiability of $f$, if in any neighborhood $v$ of $t_0$ there exist points $t_1 < t_0 < t_2$ such that:
**Type(I)**: at $t_1$(2.8) holds while (2.9) does not hold and at $t_2$(2.9) holds while (2.8) does not hold, or
**Type(II)**: at $t_1$(2.9) holds while (2.8) does not hold and at $t_2$(2.8) holds while (2.9) does not hold.

## 3. Fuzzy Impulsive Fractional Differential Equation

Consider the following fuzzy impulsive fractional differential equation

$${}_cD_*^{\frac{k}{\alpha}} y(t) = f(t, y), \quad t \in J = [0, T], \quad t \ne t_k, \ m - 1 < \frac{1}{\alpha} < m, \quad m \in \mathbb{N} \tag{3.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \tag{3.2}$$

$$y(0) = y_0, \tag{3.3}$$

where $k = 1, 2, \ldots, m, \ m - 1 < \frac{1}{\alpha} \le m$ , ${}_cD_*^{\frac{1}{\alpha}}$ denotes the Caputo fractional generalized derivative of order $\frac{1}{\alpha}$, $y(t)$ is an unknown fuzzy function of crisp variable $t$ and $f : J \times \mathbb{R}_F \to \mathbb{R}_F$, is continuous fuzzy function, $I_k : \mathbb{R}_F \to \mathbb{R}_F$, $y_0 \in \mathbb{R}_F, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta|_{t=t_k} = y(t_k^+) \ominus y(t_k^-)$, $y(t_k^+) = lim_{h \to 0^+} y(t_k + h)$, and $y(t_k^-) = lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

**Lemma 2.** *([13, 31, 32]) The initial value problem (3.1) under the conditions (3.2) and (3.3) is equivalent to one of the following integral equations:*

$$y(t) = y_0 \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s)) ds, \quad t \in [0, t_1], \tag{3.4}$$

*if* $y(t)$ *be* ${}^{cf}[(i)-gH]$*-Caputo differentiable,*

$$y(t) = y_0 \ominus (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s)) ds, \quad t \in [0, t_1], \tag{3.5}$$

*if* $y(t)$ *be* ${}^{cf}[(ii)-gH]$*-Caputo differentiable,*

$$y(t) = \begin{cases} y_0 \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s)) ds, & if\ t \in [0, t_1] \\ y_0 \ominus (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s)) ds \ominus (-1) \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s)) ds \ominus & \\ (-1) \sum_{i=1}^k I_i(y(t_i^-)), & if\ t \in (t_k, t_{k+1}] \end{cases} \tag{3.6}$$

*if there exists a point* $t_1 \in (0, t_{k+1})$ *such that* $y(t)$ *is* $[(i)-gH]$*-Caputo differentiable on* $[0, t_1]$ *and* $[(ii)-gH]$*-Caputo differentiable on* $(t_1, t_{k+1})$.

**Theorem 3.1.** *[24] Assume that*
$*(H_1)$ *There exists a constant* $0 \leq l$ *such that* $d(f(t,u), f(t,\overline{u})) \leq l d(u, \overline{u})$*, for each* $t \in [0, T]$*, and each* $u, \overline{u} \in \mathbb{R}_F$
$(H_2)$ *There exists a constant* $0 \leq l^*$ *such that* $d(I_k(u), I_k(\overline{u})) \leq l^* d(u, \overline{u})$*, for each* $u, \overline{u} \in \mathbb{R}_F$*, and* $k = 1, 2, \ldots, m$*.*
*if*

$$[\frac{T^{\frac{k}{\alpha}} l(m+1)}{\Gamma(\frac{k}{\alpha}+1)} + ml^*] < 1. \tag{3.7}$$

*Such that* $T$ *is very small numbers therefore, Eqs.(3.1)-(3.3) has a unique solution on* $[0, T]$.

## 4. Homotopy Analysis Method

Consider N[y] = 0, where N is a nonlinerar operator, $y(x, r)$ is unknown function and $x$ is an independent variable. Let $y_0$ denote an initial guess of the exact solution $y(x, r)$, $h \neq 0$ an auxiliary parameter, $H_1(x) \neq 0$ an auxiliary function, and $L$ an auxiliary linear operator with the propery $L[s(x, r)] = 0$ when $s(x, r) = 0$. Then using $q \in [0, 1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x,.q,r) - y_0(x,r)] - qhH_1(x)N[\phi(x,q,r)] = H_2[\phi(x,q,r), y_0(x,r), H_1(x), h, q] \tag{4.1}$$

It should be emphasized that we have great freedom to choose the initial guess y0(x, r), the auxiliary linear $L$, the non-zero auxiliary parameter $h$, and the auxiliary function $H_1$. Enforcing the homotopy .Eq (4.1) to be zero, i.e.

$$H_2[\phi(x,q,r), y_0(x,r), H_1(x), h, q] = 0,$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,.q,r) - y_0(x,r)] = qhH_1(x)N[\phi(x,q,r)] \tag{4.2}$$

when $q = 0$, the zero-order deformation Eq.(4.2) becomes

$$\phi(x, 0, r) = y_0(x, r) \tag{4.3}$$

and when $q = 1$, since $h \neq 0$ and $H_1 \neq 0$, the zero-order deformation Eq.(4.2) is equivalent to

$$\phi(x,1,r) = y(x,r) \tag{4.4}$$

Thus, according to Eqs.(4.3) and (4.4), as embedding parameter $q$ increases from 0 to 1, $\phi(x, q, r)$ varies continuously from the initial approximation $y_0(x, r)$ to the exact solution $y(x, r)$. Due to Taylors theorem, $\phi(x, q, r)$ can be expanded in a power series of $q$ as follows

$$\phi(x,q,r) = y_0(x,r) + \sum_{m=1}^{\infty} y_m(x,r)q^m \tag{4.5}$$

where,

$$y_m(x,r) = \frac{1}{m!}\frac{\phi(x,q,r)}{q^m}|_q = 0 \tag{4.6}$$

Let the initial guess $y_0(x, r)$, the auxiliary linear parameter $L$, the nonzero auxiliary parameter h and the auxiliary function $H_1$ be properly chosen so that the power series Eq.(4.5) of converges at $q = 1$ , then, we have under these assumptions the solution series

$$y(x,r) = \phi(x,1,r) = y_0(x,r) + \sum_{m=1}^{\infty} y_m(x,r)q^m \tag{4.7}$$

From Eq.(4.5), we can write Eq.(4.2) as follows:

$$(1-q)L[\phi(x,.q,r) - y_0(x,r)] = qhH_1(x)N[\phi(x,q,r)] = (1-q)L[\sum_{m=1}^{\infty} y_m(x,r)q^m] \tag{4.8}$$

thus,

$$L[\sum_{m=1}^{\infty} y_m(x,r)q^m] - qL[\sum_{m=1}^{\infty} y_m(x,r)q^m] = qhH_1(x)N[\phi(x,q,r)] \tag{4.9}$$

By differentiating Eq.(4.8) m-times with respect to $q$, we obtain

$$L[\sum_{m=1}^{\infty} y_m(x,r)q^m] - qL[\sum_{m=1}^{\infty} y_m(x,r)q^m]^m = qhH_1(x)N[\phi(x,q,r)]^m = m!L[y_m(x,r) - y_{m-1}(x,r)] \tag{4.10}$$

Therefore

$$L[y_m(x,r) - \chi y_{m-1}(x,r)] = hH_1(x)R_m\left(y_{m-1}(x,r)\right) \tag{4.11}$$

where,

$$R_m\left(y_{m-1}(x,r)\right) = \frac{1}{m-1}\frac{\eth^{m-1}N[\phi(x,q,r)]}{\eth q^{m-1}}|_{q=0} \tag{4.12}$$

and

$$\chi = \begin{cases} 0, & m \leqslant 1 \\ 1, & m > 1 \end{cases} \tag{4.13}$$

If we are not able to determine the sum of series in (4.7), then we can accept the partial sum of series

$$y(t,r) = \sum_{i=0}^{N} y_m(t,r). \tag{4.14}$$

for more details see ([48], [49] and [52]).

## 5. Solving Fuzzy Impulsive Fractional Differential Equation by Homotopy Analysis Method

Using Lemma (2), the solution of problems (3.1)-(3.3) is equivalent to integral Eqs. (3.4)-(3.6). We show how HPM applied to the following integral equations. Now consider the impulsive fractional deferential equation (3.6) by integration from [0, k + 1], we have:

$$y(t) = \begin{cases} y_0 \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds, t \in [0, t_1], \\ y_0 \ominus (-1)\frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus (-1) \\ \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus (-1) \sum_{i=1}^{k} I_i(y(t_i^-)), t \in (t_k, t_{k+1}], \end{cases} \tag{5.1}$$

by impulsive effect, we have:

$$\Delta y(t) = y(0^+) \ominus y(0^-) = I_0(y(0^-)), \quad \Longrightarrow y(0^+) = I_0(y(0^-)) \oplus y(0^-) \tag{5.2}$$

By substituting Eq. (5.2) into Eq. (5.1) we have

$$y(t) = \begin{cases} I_0(y(0^-)) \oplus y(0^-) \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds & \text{if} \quad t \in [0, t_1], \\ I_0(y(0^-)) \oplus y(0^-) \ominus (-1)\frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus (-1) \\ \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus (-1) \sum_{i=1}^{k} I_i(y(t_i)), t \in (t_k, t_{k+1}], \end{cases} \tag{5.3}$$

In Eqs. (5.3) we define $y(t^-) = y(t)$. Thus

$$y(t) = \begin{cases} I_0(y(0)) \oplus y(0) \oplus \frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds, t \in [0, t_1], \\ I_0(y(0)) \oplus y(0) \ominus (-1)\frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus (-1) \\ \frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds \ominus (-1) \sum_{i=1}^{k} I_i(y(t_i)), t \in (t_k, t_{k+1}], \end{cases} \tag{5.4}$$

Now operating the laplace transform on both in Eq.(5.4), we get

$$Ly(t) = \begin{cases} L[I_0(y(0)) \oplus y(0)] \oplus L\{\frac{1}{\Gamma(\frac{k}{\alpha})} \int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds\}, t \in [0, t_1], \\ L[I_0(y(0)) \oplus y(0) \ominus (-1) \sum_{i=1}^{k} I_i(y(t_i))] \ominus (-1) L[\frac{1}{\Gamma(\frac{k}{\alpha})} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\frac{k}{\alpha}-1} f(s, y(s))ds] \ominus (-1) \\ L[\frac{1}{\Gamma(\frac{k}{\alpha})} \int_{t_k}^{t} (t-s)^{\frac{k}{\alpha}-1} f(s, y(s))ds], t \in (t_k, t_{k+1}], \end{cases}$$

We define the nonlinear operator

$$N[\phi(t,q)] =$$

$$\begin{cases} L\phi(t,q) \ominus L[I_0(y(0)) \ominus y(0)] \ominus L[\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,\phi(s,q))ds], t \in [0,t_1], \\ L\phi(t,q) \ominus L[I_0(y(0)) \ominus y(0) \oplus (-1)\sum_{i=1}^{k} I_i(y(t_i))] \oplus (-1)L[\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1} f(s,\phi(s,q))ds] \oplus (-1) \\ L[\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1} f(s,\phi(s,q))ds], t \in (t_k, t_{k+1}], \end{cases} \quad (5.5)$$

So where $q \in [0, 1]$ be an embedding parameter and $\phi(t, q)$ is real function of $x$ and $q$. By means of traditional Homotopy method the great mathematician Liao [52] construct the zero order deformation equation

$$(1-q)L[\phi(t,q) - y_0(x,r)] = hqH_1(x)N\Big(\phi(t,q)\Big) \quad (5.6)$$

Where $h$ is nonzero parameter, $H(t) \neq 0$, $y_0(t)$ is an inital guess of $y(t)$ and $\phi(t, q)$ is un known function. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t,0) = y_0(t,r), \quad \phi(t,1) = y(t,r) \quad (5.7)$$

We take an initial guess

$$y_0(t) = \begin{cases} I_0(y(0)) \oplus y(0), & \text{if} \quad t \in [0,t_1], \\ I_0(y(0)) \oplus y(0) \ominus (-1)\sum_{i=1}^{k} I_i(y(t_i)), & \text{if} \quad t \in (t_1, t_{k+1}], \end{cases} \quad (5.8)$$

and

$$y_m(t,r) =$$

$$\begin{cases} \chi_m y_{m-1}(t,r) + L^{-1}hH_1(x)\big[y_{m-1}(t) \ominus \frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,y_{m-1}(s))ds \\ \ominus(1-\chi_m)[I_0(y_{m-1}(0)) \oplus y_{m-1}(0)]\big], t \in [0,t_1], \\ \chi_m y_{m-1}(t,r) + L^{-1}hH_1(x)\big[y_{m-1}(t) \oplus (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1} f(s,y_{m-1}(s))ds \\ \oplus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1} f(s,y_{m-1}(s))ds \ominus \\ (1-\chi_m)[I_0(y_{m-1}(0)) \oplus y_{m-1}(0) \ominus (-1)\sum_{i=1}^{k} I_i(y_{m-1}(t_i^-))]\big], t \in (t_k, t_{k+1}], \end{cases} \quad (5.9)$$

We approximate $\underline{y}(t,r) = [\lim_{m\to\infty}(\underline{y_m}(t,r))]$ and $\overline{y}(t,r) = [\lim_{m\to\infty}(\overline{y_m}(t,r))]$. We say that $u$ is $[(i)-gH]$-Caputo differentiable at $t$ if.

$$y(t,r) = [\underline{y}(t,r), \overline{y}(t,r)]$$

$y$ is $[(ii)-gH]$-Caputo differentiable at $t$ if.

$$y(t,r) = [\overline{y}(t,r), \underline{y}(t,r)]$$

The relations above have been obtained with the assumption of the convergence of series (4.7).The conditions for such convergence are discussed in the following theorem:

**Theorem 5.1.** *Let the functions $K = (t-s)^{\frac{k}{\alpha}-1}$ and $f = f(s, y_n(s))$ , appearing in Eq.(5.4), be continuous in the respective domains, i.e. $K, f \in C([a,b] \times [a,b])$. If additionally the following inequality:*

$$|\lambda| M_1 < \frac{1}{b-a} \quad (5.10)$$

*is satisfied and as the initial approximation y0, a function continuous in the interval [a, b] is chosen, then series (4.7), in which the functions $y_{n+1}$ are determined by means of relations (5.8)-(5.9), is uniformly convergent in the interval [a, b] for each p $\in$ [0, 1].*

**Proof.** Certainly, *K* and *f* are bounded; this means that there exist the positive numbers M1 and N1 such that

$$|K(x,t)| \leq M1 \qquad ,|I(y_0(s))| \leq N1 \qquad \forall x,t \in [a,b]. \tag{5.11}$$

Let $y_0 \in C[a, b]$. Therefore there exists a positive number $N_0$ such that

$$|y_0(x)| \leq N_0 \qquad \forall x \in [a,b]. \tag{5.12}$$

The assumptions made imply the following estimations:

$$|y_0(x)| = \begin{cases} |I_0(y(0)) \oplus y(0)| \leq N_0 + N_1, & \text{if} \quad t \in [0,t_1], \\ |I_0(y(0)) \oplus y(0) \ominus_{gH} (-1)\sum_{i=1}^{k} I_i(y(t_i)| \leq N_0 + N_1 + N_2, & \text{if} \quad t \in (t_1, t_{k+1}], \end{cases} \tag{5.13}$$

$$|y_1(x)| = \begin{cases} \left|\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t (t-s)^{\frac{k}{\alpha}-1} f(s,y_0(s))ds\right| \leq \frac{N_0(t-0)|\lambda|M}{\Gamma(\frac{k}{\alpha})}, \text{if} \quad t \in [0,t_1], \\ \Big| \ominus (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1} f(s,y_0(s))ds \ominus \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1} f(s,y_0(s))ds\Big| \leq \frac{N_0|\lambda|M}{\Gamma(\frac{k}{\alpha})}\Big(\sum_{i=1}^{k}(t_i - t_{i-1}) + (t-t_k)\Big), \quad \text{if} \quad t \in (t_1, t_{k+1}], \end{cases} \tag{5.14}$$

where $B := N_0 + N_1 + N_0(b-a)(|\lambda|M1)$. In general we have

$$|y_{n+1}(x)| \leq B(b-a)^{m-1}(|\lambda|M1)^{m-1} \quad ,x \in [a,b]. \tag{5.15}$$

In this way, for the series considered, (4.7) we get, for $q \in [0, 1]$,

$$|\sum_{n=0}^{\infty} q^m u_m| \leq \sum_{m=0}^{\infty} |y_m| \leq a_0 + \sum_{m=1}^{\infty} a_m \tag{5.16}$$

Where

$$a_0 = N_0 + N_1, a_m = B(b-a)^{m-1}(|\lambda|M1)^{m-1} \tag{5.17}$$

The last series in the above estimation is the convergent geometric series possessing the common ratio $q = (|\lambda|M)(b-a) < 1$ . Hence, series considered, (4.7), is uniformly convergent in the interval [a, b] for each $q \in [0, 1]$.

**Theorem 5.2.** The exact solution of Eqs (3.1) − (3.3) could be represented by

$$y(t) = \begin{cases} y_0 + \sum_{n=1}^{\infty} y_m(t), & if \quad t \in [0,t_1], \\ y_0 + \sum_{m=1}^{\infty} y_m(t), & if \quad t \in (t_1, t_{k+1}], \end{cases} \tag{5.18}$$

Also, the approximate solution $y(t)$ can be obtained by taking finitely many terms in the series representation of $y(t)$ so,

$$\widehat{y}(t) = \begin{cases} \sum_{m=0}^{N} y_m(t), & if \quad t \in [0,t_1], \\ \sum_{m=0}^{N} y_m(t), & if \quad t \in (t_1, t_{k+1}], \end{cases} \tag{5.19}$$

And $\hat{y}$(t) convergence uniformly to the exact solution $y(t)$.

**Proof:** Let $y(t)$ be solution of Eqs.(3.1) − (3.3). From Eqs. (5.11) and (5.12). $y(t)$ could be expressed by series as follow

$$y(t)=\begin{cases}\sum_{n=1}^{\infty} y_m(t), & \text{if} \quad t\in[0,t_1],\\ \sum_{n=1}^{\infty} y_m(t), & \text{if} \quad t\in(t_1,t_{k+1}],\end{cases} \tag{5.20}$$

Substituting Eq. (5.8) into Eq. (5.20), where

$$y_0(t)=\begin{cases}I_0(y(0))\oplus u(0), & \text{if} \quad t\in[0,t_1],\\ I_0(y(0))\oplus y(0)\ominus(-1)\sum_{i=1}^{k} I_i(y(t_i)), & \text{if} \quad t\in(t_1,t_{k+1}],\end{cases} \tag{5.21}$$

And

$$y_m=\begin{cases}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,y_{m-1}(s))ds, & \text{if} \quad t\in[0,t_1],\\ \ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,y_{m-1}(s))ds\ominus & \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,y_{m-1}(s))ds, & \text{if} \quad t\in(t_1,t_{k+1}],\end{cases} \tag{5.22}$$

Also let us $y(t)$ and $\hat{y}(t)$ be exact and approximate solution of the problems (3.1) − (3.3). We will show that,

$$d(y(t),\widehat{y}(t))\le\sum_{n=1}^{N}||d(u_{m-1}(t),\widehat{y_{m-1}}(t))||_\alpha \tag{5.23}$$

Therefore if $y(t)$ and $\hat{y}(t)$ be exact and approximate solution of the problems (3.1) − (3.3), without loss of generality suppose that there exists a point $t_1 \in (0, t_k+1)$ such that $y(t)$ is $[(i) - gH]$-Caputo differentiable on $[0, t_1]$ and $[(ii) - gH]$-Caputo differentiable on $(t_1, t_k+1)$. Using Lemma (2), the solution of problems (3.1) − (3.3) in this case is equivalent to integral Eq. (3.6). By using Lemma 2 we have,

$$y(t)=$$

$$\begin{cases}\sum_{m=1}^{\infty}\left(\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,u_{m-1}(s))ds\right), \ t\in[0,t_1]\\ \sum_{m=1}^{\infty}\Big(\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,u_{m-1}(s))ds\\ \ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,u_{m-1}(s))ds\ominus\Big), t\in(t_k,t_{k+1}]\end{cases} \tag{5.24}$$

And

$$\widehat{y}(t)=$$

$$\begin{cases}\sum_{m=1}^{N}\left(\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{y_{m-1}}(s))ds\right), \ t\in[0,t_1],\\ \sum_{m=1}^{N}\Big(\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,\widehat{y_{m-1}}(s))ds\\ \ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{y_{m-1}}(s))ds\Big), t\in(t_k,t_{k+1}]\end{cases} \tag{5.25}$$

thus

$$d(y(t),\widehat{y}(t))=$$

$$\begin{cases}d\Big(\sum_{m=1}^{\infty}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,y_{m-1}(s))ds,\sum_{m=1}^{N}\frac{1}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{y_{m-1}}(s))ds\Big)\le\\ |\sum_{m=1}^{N}\frac{l}{\Gamma(\frac{k}{\alpha})}\int_0^t(t-s)^{\frac{k}{\alpha}-1}d(y_{m-1}(s),\widehat{y_{m-1}}(s))ds|\le\\ \sum_{m=1}^{N}\frac{l}{\Gamma(\frac{k}{\alpha})}\Big(\int_0^t(t-s)^{\frac{k-\alpha}{\alpha}}ds\Big)\Big(\int_0^t(d(y_{m-1}(s),y_{m-1}N(s)))ds)^\alpha\Big)^{\frac{1}{\alpha}}=\\ \sum_{m=1}^{N}\frac{l}{\Gamma(\frac{k}{\alpha})}\Big(\frac{\Gamma(\frac{k}{\alpha})T^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)\Big(\int_0^t(d(y_{m-1}(s),\widehat{y_{m-1}}(s)))ds)^\alpha\Big)^{\frac{1}{\alpha}}\\ =\sum_{m=1}^{N}\frac{lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}||d(y_{m-1}(t),\widehat{y_{m-1}}(t))||_\alpha, \ t\in[0,t_1]\end{cases} \tag{5.26}$$

$$d(y(t),\widehat{y}(t)) =$$
$$\begin{cases} d\Big(\ominus(-1)\sum_{m=1}^{\infty}\Big(\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,y_{m-1}(s))ds\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,y_{m-1}(s))ds\ominus\Big), \\ \ominus(-1)\sum_{m=1}^{N}\Big(\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}f(s,\widehat{y_{m-1}}(s))ds\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}f(s,\widehat{y_{m-1}}(s))ds\Big)\Big) \\ \le d\Big(\sum_{m=1}^{N}\Big(\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}d(f(s,y_{m-1}(s)),f(s,\widehat{u_{m-1}}(s)))ds\ominus \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}d(f(s,y_{m-1}(s)),f(s,\widehat{y_{m-1}}(s)))ds\Big)\Big) \\ \le d\Big(\sum_{n=1}^{N}\Big(\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}d(f(s,y_{m-1}(s)),f(s,\widehat{y_{m-1}}(s)))ds\ominus \\ (-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}d(f(s,y_{m-1}(s)),f(s,\widehat{y_{m-1}}(s)))ds\Big)\Big) \\ \le \sum_{m=1}^{N}\Big(\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\frac{k}{\alpha}-1}ld(y_{m-1}(s),\widehat{y_{m-1}}(s))ds\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\int_{t_k}^{t}(t-s)^{\frac{k}{\alpha}-1}ld(y_{m-1}(s),\widehat{y_{m-1}}(s))ds\Big) = \\ \ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\sum_{i=1}^{k}\Big(\frac{\Gamma(\frac{k}{\alpha})lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)\Big(\int_0^t(d(y_{m-1}(s),\widehat{y_{m-1}}(s))ds)^{\alpha}\Big)^{\frac{1}{\alpha}}\ominus(-1)\frac{1}{\Gamma(\frac{k}{\alpha})}\Big(\frac{\Gamma(\frac{k}{\alpha})lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)\Big(\int_0^t(d(y_{m-1}(s),\widehat{y_{m-1}}(s))ds)^{\alpha}\Big)^{\frac{1}{\alpha}} \\ = \sum_{m=1}^{N}\Big(\ominus(-1)\frac{MlT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}||y_{m-1}(t),\widehat{y_{m-1}}(t)||_{\alpha}\ominus_{gH}(-1)\frac{MlT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}||y_{m-1}(t),\widehat{y_{m-1}}(t)||_{\alpha}\Big) \\ = \Big(\Big(\frac{MlT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}+\frac{lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}+\Big)||y_{m-1}(t),\widehat{y_{m-1}}(t)||_{\alpha}\Big) = \sum_{m=1}^{N}\Big(\Big(\frac{(M+1)lT^{\frac{k}{\alpha}}}{\Gamma(\frac{k}{\alpha}+1)}\Big)||y_{m-1}(t),\widehat{y_{m-1}}(t)||_{\alpha}\Big),\ t\in(t_k,t_{k+1}] \end{cases} \tag{5.27}$$

It follows that

$$d(y(t),\widehat{y}(t)) \le \sum_{m=1}^{N}||d(y_{m-1}(t),\widehat{y_{m-1}}(t))||_{\alpha}$$

where $M_1$ and $M_2$ are constants. Hence $\|d(y_{m-1}(t),\widehat{y_{m-1}}(t)\|_0 \to 0$ as $m \to \infty$, the approximate solution convergence uniformly to the exact solution $y(t)$ . The proof for other case is similar.

## 6. Numerical Example

We demonstrate the effectiveness of the Homotopy Analysis Method, for solving fuzzy impulsive fractional differential equations by the following some examples.

**Example 1.** Let us consider the fuzzy impulsive fractional equation,

$$ {}_cD_*^{\frac{k}{\alpha}}y(t) = \frac{ty^2(t)}{(3+t)(1+y^2(t))},\quad t\in J:=[0,1],\quad t\neq\frac{1}{2},\quad m-1<\frac{1}{\alpha}<m,\quad m\in N\ , \tag{6.1}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1^-}{2})|}{2+|y(\frac{1^-}{2})|}, \tag{6.2}$$

$$y(0)=[\tilde{0},\tilde{0}] \tag{6.3}$$

$$Y_{\frac{1}{\alpha}}(0)=[\tilde{0},\tilde{0}] \tag{6.4}$$

Set

$$I_k(t)=\frac{t}{t+2}, t\in[0,\infty) \tag{6.5}$$

$$f(t,y(t))=\Big[\frac{t^3(r-1)}{(3+t)(1+t^2)},\frac{t^3(1-r)}{(3+t)(1+t^2)}\Big] \tag{6.6}$$

Suppose $y(t)$ is $[(i)-gH]$-Caputo differentiable in [0, 1] and $[(ii)-gH]$-Caputo differentiable in(1, 2] proof of the other cases are left to the reader. Using the Eqs (5.8)-(5.9) and taking k = 1 and $\frac{1}{a}$ =

$\frac{1}{4}$. With initial conditions $y(0) = [\tilde{0}, \tilde{0}]$ . To solve the equation (6.1) by means of Homotopy Analysis Method, according to the initial conditions denoted in Eqs. (6.2)-(6.6), it is natural to choose.

$$y_0(t) = \begin{cases} I_0(y(0)) \oplus y(0), & \text{if} \quad t \in [0,1], \\ I_0(y(0)) \oplus y(0) \ominus (-1)I_1(y(t_1)), & \text{if} \quad t \in (1,2], \end{cases} \tag{6.7}$$

So

$$y_0(t) = \begin{cases} \frac{1}{5}, & \text{if} \quad t \in [0,1], \\ \frac{8}{15}, & \text{if} \quad t \in (1,2], \end{cases} \tag{6.8}$$

$$y(t,r) =$$

$$\begin{cases} \frac{1}{5} \ominus \frac{1}{24} \int_0^t (t-s)^{\frac{k}{\alpha}-1} \left[\frac{s^3(r-1)}{(3+s)(1+s^2)}, \frac{s^3(1-r)}{(3+s)(1+s^2)}\right] ds, t \in [0,1], \\ \frac{1}{5} \oplus (-1)\frac{1}{24} \int_0^1 (1-s)^{\frac{k}{\alpha}-1} \left[\frac{s^3(r-1)}{(3+s)(1+s^2)}, \frac{s^3(1-r)}{(3+s)(1+s^2)}\right] ds \oplus (-1) \\ \frac{1}{24} \int_1^t (t-s)^{\frac{1}{\alpha}-1} \left[\frac{s^3(r-1)}{(3+s)(1+s^2)}, \frac{s^3(1-r)}{(3+s)(1+s^2)}\right] ds \oplus (-1)\frac{1}{3}, t \in (1,2], \end{cases} \tag{6.9}$$

Applying the Laplace transform on both dies in Eq.(6.9) and after using convolution property of Laplace transform, we get

$$Ly(t,r) =$$

$$\begin{cases} L[\frac{1}{5}] \ominus L[\frac{1}{24} \int_0^t (t-s)^{\frac{k}{\alpha}-1} \left[\frac{s^3(r-1)}{(3+s)(1+s^2)}, \frac{s^3(1-r)}{(3+s)(1+s^2)}\right]] ds, t \in [0,1], \\ L[\frac{1}{5}] \oplus (-1)L[\frac{1}{24} \int_0^1 (1-s)^{\frac{k}{\alpha}-1} \left[\frac{s^3(r-1)}{(3+s)(1+s^2)}, \frac{s^3(1-r)}{(3+s)(1+s^2)}\right] ds] \oplus (-1) \\ L[\frac{1}{24} \int_1^t (t-s)^{\frac{1}{\alpha}-1} \left[\frac{s^3(r-1)}{(3+s)(1+s^2)}, \frac{s^3(1-r)}{(3+s)(1+s^2)}\right] ds] \oplus (-1)L[\frac{1}{3}], t \in (1,2], \end{cases} \tag{6.10}$$

We choose the linear operator

$$\ell[\phi(x,q)] = L\phi(x,q) \tag{6.11}$$

With property $\ell(c) = 0$, c is constant. We now define a nonlinear operator as

$$N[\phi(x,q)] =$$

$$\begin{cases} L\phi(x,q) \ominus L[\frac{1}{5}] \oplus \frac{1}{24S^2} L\phi(x,q), t \in [0,1], \\ L\phi(x,q) \ominus L[\frac{1}{5}] \oplus (-1)\frac{1}{24S^2} L\phi(x,q) \oplus (-1) \\ \frac{1}{24S^2} L\phi(x,q) \oplus (-1)L[\frac{1}{3}], t \in (1,2], \end{cases} \tag{6.12}$$

Using above definition, with assumption $H(t) = 1$,we construct the zero order deformation equation

$$(1-q)L[\phi(t,q) - u_0(x,r)] = hqH_1(x)N\Big(\phi(t,q)\Big) \tag{6.13}$$

Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t,0) = u_0(t,r), \quad \phi(t,1) = u(t,r) \tag{6.14}$$

Thus, we obtain the $m$th order deformation equation

$$L[y_m(x,r) - \chi y_{m-1}(x,r)] = hH_1(x)R_m\Big(y_{m-1}(x,r)\Big) \tag{6.15}$$

Operating the invers Laplace transform, we have

$$y_m(x,r) = \chi y_{m-1}(x,r)] + L^{-1}hqH_1(x)R_m\Big(y_{m-1}(x,r)\Big) \tag{6.16}$$

Where

$$R_m[y_{m-1}(t,r)] =$$
$$\begin{cases} Ly_{m-1}(t) \ominus L[I_0(y_{m-1}(0)) \ominus_{gH} y_{m-1}(0)] \ominus_{gH} \frac{1}{24S^2} L(y_{m-1}(t)), t \in [0,1], \\ Ly_{m-1}(t) \ominus L[I_0(y_{m-1}(0)) \ominus u_{m-1}(0)] \oplus (-1)\frac{1}{24} \int_0^1 (1-s)^{\frac{k}{\alpha}-1} f(s, y_{m-1}(s))ds \\ \oplus(-1)\frac{1}{24S^2} L(y_{m-1}) oplus(-1) L[I_1(y_{m-1}(1))], t \in (1,2], \end{cases} \quad (6.17)$$

The results are shown in Table 1 and Table 2.

**Table 1:** Numerical results of Example 1 for $y_{i.gH}(t)$ and $y_{ii.gH}(t)$

| $r/t$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 0.1 | 1.2 | 1.40 | 1.6 | 1.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0 | 0.008612 | 0.048718 | 0.134252 | 0.275593 | 6.472847 | 6.739047 | 7.088843 | 7.529535 | 8.067877 |
| 0.2 | 0 | 0.003163 | 0.017894 | 0.049310 | 0.101224 | 1.353031 | 1.619232 | 1.969028 | 2.40972 | 2.948062 |
| 0.3 | 0 | 0.001155 | 0.006534 | 0.018005 | 0.036959 | 0.669863 | 0.936063 | 1.285859 | 1.72655 | 2.264893 |
| 0.4 | 0 | 0.00031 | 0.001751 | 0.004825 | 0.009905 | 0.470790 | 0.736991 | 1.086787 | 1.527471 | 2.065821 |
| 0.5 | 0 | 3.30E-05 | 0.000187 | 0.000514 | 0.001055 | 0.421958 | 0.688159 | 1.037954 | 1.478646 | 2.016988 |
| 0.6 | 0 | -1.49E-05 | -8.40E-05 | -0.000231 | -0.000475 | 0.413813 | 0.680014 | 1.029810 | 1.470502 | 2.008844 |
| 0.7 | 0 | -6.24E-05 | -0.000353 | -0.000972 | -0.001995 | 0.403001 | 0.669202 | 1.018998 | 1.459690 | 1.998032 |
| 0.8 | 0 | -9.53E-05 | -0.000540 | -0.001486 | -0.003051 | 0.395892 | 0.662093 | 1.011889 | 1.452581 | 1.990923 |
| 0.9 | 0 | -7.74E-05 | -0.000438 | -0.001206 | -0.002476 | 0.400272 | 0.666472 | 1.016268 | 1.456960 | 1.995302 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0.418116 | 0.684316 | 1.034113 | 1.474805 | 2.013146 |

**Table 2:** Numerical results of Example 1 for $\overline{y_{i.gH}}(t)$ and $\overline{y_{ii.gH}}(t)$

| $r/t$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 0.1 | 1.2 | 1.40 | 1.6 | 1.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.059254 | 0.093470 | 0.13742 | 0.191876 | 0.257574 | 0.661318 | 0.927518 | 1.277314 | 1.718006 | 2.256348 |
| 0.2 | 0.03893 | 0.061417 | 0.090293 | 0.126076 | 0.169244 | 0.578712 | 0.844913 | 1.19471 | 1.635401 | 2.173743 |
| 0.3 | 0.021318 | 0.033627 | 0.049438 | 0.069030 | 0.092665 | 0.507010 | 0.773210 | 1.123007 | 1.563699 | 2.102041 |
| 0.4 | 0.008134 | 0.012831 | 0.018864 | 0.026339 | 0.035358 | 0.453387 | 0.719584 | 1.069380 | 1.510072 | 2.048414 |
| 0.5 | 0.001245 | 0.001964 | 0.002888 | 0.004032 | 0.005413 | 0.424570 | 0.6907706 | 1.040567 | 1.481259 | 2.019601 |
| 0.6 | 0.001229 | -0.0019395 | -0.002850 | -0.003980 | -0.005342 | 0.415555 | 0.681755 | 1.031551 | 1.472243 | 2.010585 |
| 0.7 | -0.007416 | -0.011698 | -0.017198 | -0.024014 | -0.032236 | 0.395541 | 0.661742 | 1.011538 | 1.452230 | 1.990572 |
| 0.8 | -0.016141 | -0.025462 | -0.037433 | -0.052268 | -0.070165 | 0.355179 | 0.621380 | 0.971176 | 1.411868 | 1.950210 |
| 0.9 | -0.019648 | -0.030994 | -0.045566 | -0.063624 | -0.085408 | 0.314236 | 0.580437 | 0.930233 | 1.370925 | 1.909267 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0.418116 | 0.684316 | 1.034113 | 1.474805 | 2.013146 |

## 7. Conclusions

In this paper a new approach by presenting Homotopy Analysis Method expansion based on gH-differentiability was introduced. The numerical results show that the present method is an accurate and reliable analytical technique for fuzzy impulsive fractional differential equation. All numerical results are obtained using software Matlab.

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