



A Subclass of Close-to-Star Functions

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Article history: Received 5 November 2012, Received in revised form 12 December 2012, Accepted 14 December 2012, Published 17 December 2012.

Abstract: This paper is concerned with a subclass of close to star functions in the unit disc E . We obtain the Coefficient bounds, distortion theorem, argument theorem and some invariance properties for the functions belonging to this class.

Keywords: Subordination, Analytic functions, Starlike functions, Close-to-Star functions.

Mathematics Subject Classifications: 30C45

1. Introduction

Let U be the class of Schwarzian functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1, \quad z \in E.$$

Let A denote the class of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analytic in the unit disc E .

For functions f and g analytic in E , we say that f is subordinate to g , denoted by $f \prec g$, if there exists a Schwarz function $w(z) \in U$ such that $f(z) = g(w(z))$.

For C and D , $-1 \leq D < C \leq 1$, a function $p(z)$ analytic in E with $p(0) = 1$, is in the class $P(C, D)$ if $p(z) \prec \frac{1+Cz}{1+Dz}$. This class was introduced by Janowski [5].

Given A and B ($-1 \leq B < A \leq 1$), $S^*(A, B)$ denote the class of functions f analytic in E with $f(0) = f'(0) - 1 = 0$ such that $\frac{zf'(z)}{f(z)} \in P(A, B)$. The class $S^*(A, B)$ was introduced by Janowski [5] and studied further by Goel and Mehrok ([1], [2]). Also $S^*(1, -1) \equiv S^*$, the well-known class of starlike functions.

A function $f(z) \in A$ is said to be in the class $S(A, B)$ if it satisfies the following condition

$$\frac{f(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1, z \in E, g(z) \in S^*.$$

The class $S(A, B)$ was introduced and studied by Mehrok et al. in [6]. Also $S(1, -1) \equiv S$, the class of close-to-star functions introduced by Reade [7].

A function $f(z) \in A$ is said to be in the class $S(A, B; C, D)$, $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1$ if there exists $g \in S^*(A, B)$ such that $\frac{f(z)}{g(z)} \in P(C, D)$.

The following observations are obvious:

- (i) $S(1, -1; 1, -1) \equiv S$.
- (ii) $S(A, B; 1, -1) \equiv S(A, B)$.

In the present paper, we study the class $S(A, B; C, D)$ and obtain coefficient bounds, distortion theorem, argument theorem and some invariance properties.

2. Coefficient Bounds

Unless otherwise mentioned in the sequel, the only restrictions on the real constants A, B, C and D are that $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$.

Lemma A[4] Let

$$p(z) = \frac{1 + Az}{1 + Bz} = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

then $|p_n| \leq (A - B), n \geq 1.$

The bounds are sharp, being attained for the functions

$$P(z) = \frac{1 + A\delta z^n}{1 + B\delta z^n}, |\delta| = 1.$$

Lemma B[1] If $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^*(A, B)$, then

$$|b_2| \leq (A - B).$$

$$|b_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^n (A - (j-1)B), \quad A - (n-1)B \geq (n-2), \quad n \geq 3.$$

Theorem 1. If $f(z) \in A$, then for $A - (n-1)B \geq (n-2)$, $n \geq 2$,

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^n (A - (j-1)B) + (C - D) \left[1 + \sum_{k=2}^{n-1} \frac{1}{(k-1)!} \prod_{j=2}^k (A - (j-1)B) \right]. \tag{2.1}$$

The bounds are sharp.

Proof. Since $f \in S(A, B; C, D)$, there exists a function $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^*(A, B)$ and a Schwarz function $w(z)$ such that

$$\frac{f(z)}{g(z)} = \frac{1 + Cw(z)}{1 + Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k, \quad z \in E. \tag{2.2}$$

From (2.2), it yields

$$z + \sum_{k=2}^{\infty} a_k z^k = \left(z + \sum_{k=2}^{\infty} b_k z^k \right) \left(1 + \sum_{k=1}^{\infty} p_k z^k \right). \tag{2.3}$$

On equating the coefficients of z^n in (2.3), we have

$$a_n = b_n + p_1 b_{n-1} + p_2 b_{n-2} + \dots + p_{n-1}. \tag{2.4}$$

Using Lemma A, (2.4) becomes

$$|a_n| \leq |b_n| + (C - D) [|b_{n-1}| + |b_{n-2}| + \dots + |b_2| + 1]. \tag{2.5}$$

(2.5) together with Lemma B, yields

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^n (A - (j-1)B) + (C - D) \left[\frac{1}{(n-2)!} \prod_{j=2}^{n-1} (A - (j-1)B) + \frac{1}{(n-3)!} \prod_{j=2}^{n-2} (A - (j-1)B) + \dots + (A - B) + 1 \right]$$

which on simplification, yields (2.1). That completes the proof.

The function $f_0(z)$ defined by,

$$f_0(z) = \begin{cases} \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right) z(1+B\delta_2 z)^{\frac{(A-B)}{B}}; B \neq 0 \\ \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right) z \exp(A\delta_2 z); B = 0, |\delta_1| = |\delta_2| = 1, \end{cases} \tag{2.6}$$

shows that the bound (2.1) is sharp for each $n \geq 2$.

For $A = C = 1$ and $B = D = -1$, Theorem 1 agrees with the following result of Reade [7].

Corollary. If f is a close-to-star function, then $|a_n| \leq n^2, n \geq 2$.

3. Distortion Theorem

Lemma C[5] If $g \in S^*(A, B)$, then for $|z| = r < 1$,

$$r(1 - Br)^{\frac{(A-B)}{B}} \leq |g(z)| \leq r(1 + Br)^{\frac{(A-B)}{B}}, B \neq 0;$$

$$r \exp(-Ar) \leq |g(z)| \leq r \exp(Ar), B = 0.$$

Theorem 2. For $f \in S(A, B; C, D)$ and for $|z| = r, 0 < r < 1$, we have

$$r \left(\frac{1 - Cr}{1 - Dr}\right) (1 - Br)^{\frac{(A-B)}{B}} \leq |f(z)| \leq r \left(\frac{1 + Cr}{1 + Dr}\right) (1 + Br)^{\frac{(A-B)}{B}}, B \neq 0; \tag{3.1}$$

$$r \left(\frac{1 - Cr}{1 - Dr}\right) \exp(-Ar) \leq |f(z)| \leq r \left(\frac{1 + Cr}{1 + Dr}\right) \exp(Ar), B = 0. \tag{3.2}$$

Estimates are sharp.

Proof. From (2.2), we have

$$|f(z)| = |g(z)| \left| \frac{1 + Cw(z)}{1 + Dw(z)} \right|.$$

It is easy to show that

$$\frac{1 - Cr}{1 - Dr} \leq \left| \frac{1 + Cw(z)}{1 + Dw(z)} \right| \leq \frac{1 + Cr}{1 + Dr}. \tag{3.3}$$

Since $g \in S^*(A, B)$, thus due to Lemma C, the result is obvious.

Equality signs in (3.1) and (3.2) are attained by the function defined by (2.6).

4. Argument Theorem

Lemma D[3]. If $g \in S^*(A, B)$, then for $|z| = r < 1$,

$$\left| \arg \frac{g(z)}{z} \right| \leq \frac{(A-B)}{B} \sin^{-1}(Br), B \neq 0;$$

$$\left| \arg \frac{g(z)}{z} \right| \leq Ar, B = 0.$$

Theorem 3. For $f \in S(A, B; C, D)$,

$$\left| \arg \frac{f(z)}{z} \right| \leq \frac{(A-B)}{B} \sin^{-1}(Br) + \sin^{-1} \frac{(C-D)r}{1-CDr^2}, B \neq 0; \tag{4.1}$$

$$\left| \arg \frac{f(z)}{z} \right| \leq Ar + \sin^{-1} \frac{(C-D)r}{1-CDr^2}, B = 0. \tag{4.2}$$

These inequalities are sharp.

Proof. It is easy to show that, the transformation $\frac{f(z)}{g(z)} = \frac{1+Cw(z)}{1+Dw(z)}$ maps $|w(z)| \leq r$ onto the circle

$$\left| \frac{f(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2} \right| \leq \frac{(C-D)r}{(1-D^2r^2)}, |z| = r.$$

Therefore

$$\left| \arg \frac{f(z)}{g(z)} \right| \leq \sin^{-1} \frac{(C-D)r}{1-CDr^2}. \tag{4.3}$$

Since

$$\left| \arg \frac{f(z)}{z} \right| \leq \left| \arg \frac{f(z)}{g(z)} \right| + \left| \arg \frac{g(z)}{z} \right|. \tag{4.4}$$

Using (4.3) and Lemma D in (4.4), we obtain (4.1) and (4.2). Equality signs in (4.1) and (4.2) hold for the functions $f_1(z)$ and $f_2(z)$, respectively, where

$$f_1(z) = \left(\frac{1+C\delta_1 z}{1+D\delta_1 z} \right) z (1+B\delta_2 z)^{\frac{(A-B)}{B}},$$

$$f_2(z) = \left(\frac{1+C\delta_1 z}{1+D\delta_1 z} \right) z \exp(A\delta_2 z),$$

and $\delta_1 = \frac{ir}{z}, \delta_2 = \frac{r}{z} \left[-Br + i(1-B^2r^2)^{1/2} \right].$

5. Invariance Properties

The Convolution or Hadamard Product of two functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$

is denoted by $f * g$ and defined as $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$.

Lemma E[9]. Let ϕ be convex and g be starlike in E . Then for F analytic in E with $F(0) = 1$,

$\frac{\phi * Fg}{\phi * g}(E)$ is contained in the convex hull of $F(E)$.

Lemma F[10]. If $g \in S^*(A, B)$, then for ϕ convex, $\phi * g \in S^*(A, B)$.

Theorem 4. If ϕ is convex and $f \in S(A, B; C, D)$, then $\phi * f \in S(A, B; C, D)$.

Proof. For $f \in S(A, B; C, D)$, there exists $g \in S^*(A, B)$ and $F \in P(C, D)$ such that

$$f(z) = g(z)F(z).$$

Since $\frac{1+Cz}{1+Dz}$ is convex in E , by Lemma E

$$\frac{\phi * f}{\phi * g} = \frac{\phi * Fg}{\phi * g} \prec \frac{1+Cz}{1+Dz} \quad (z \in E) \tag{5.1}$$

ϕ being a convex function.

From Lemma F, $\phi * g \in S^*(A, B)$. Thus (5.1) is equivalent to $\phi * f \in S(A, B; C, D)$.

Corollary. If $f \in S(A, B; C, D)$, then so are

$$(i) \quad F_1(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \text{Re } \gamma > 0$$

and

$$(ii) \quad F_2(z) = \int_0^z \frac{f(t) - f(st)}{t - st} dt, |s| \leq 1, s \neq 1.$$

Proof. Observe that $F_j(z) = (h_j * f)$, $j = 1, 2$, where

$$h_1(z) = \sum_{k=1}^{\infty} \frac{1+\gamma}{k+\gamma} z^k, \text{Re } \gamma > 0 \text{ and } h_2(z) = \sum_{k=1}^{\infty} \frac{1-s^k}{(1-s)k} z^k = \frac{1}{1-s} \log\left(\frac{1-sz}{1-z}\right), |s| \leq 1, s \neq 1.$$

Since due to [4], h_1 is proved to be convex, thus h_2 also is convex.

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