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Article

A Subclass of Close-to-Star Functions

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Abstract: This paper is concerned with a subclass of close to star functions in the unit disc *E*. We obtain the Coefficient bounds, distortion theorem, argument theorem and some invariance properties for the functions belonging to this class.

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1. Introduction

Let U be the class of Schwarzian functions

$$w(z)=\sum_{k=1}^{\infty}c_k z^k$$

which are analytic in the unit disc $E=\{z:|z|<1\}$ and satisfying the conditions

$$w(0)=0 \text{ and } |w(z)|<1,\ z\in E.$$

Let A denote the class of functions $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$, analytic in the unit disc *E*.

For functions f and g analytic in *E*, we say that f is subordinate to g, denoted by $f\prec g$, if there exists a Schwarz function $w(z)\in U$ such that $f(z)=g(w(z))$.

For C and D, $-1\le D<C\le 1$, a function $p(z)$ analytic in E with $p(0)=1$, is in the class $P(C,D)$ if $p(z)\prec\frac{1+Cz}{1+Dz}$. This class was introduced by Janowski [5].

Given A and B $(-1 \le B < A \le 1)$, $S^*(A,B)$ denote the class of functions f analytic in E with $f(0) = f'(0) - 1 = 0$ such that $\frac{zf'(z)}{f(z)} \in P(A,B)$. The class $S^*(A,B)$ was introduced by Janowski [5] and studied further by Goel and Mehrok ([1], [2]). Also $S^*(1,-1) \equiv S^*$, the well-known class of starlike functions.

A function $f(z) \in A$ is said to be in the class $S(A,B)$ if it satisfies the following condition

$$\frac{f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in E, \; g(z) \in S^*.$$

The class $S(A,B)$ was introduced and studied by Mehrok et al. in [6]. Also $S(1,-1) \equiv S$, the class of close-to-star functions introduced by Reade [7].

A function $f(z) \in A$ is said to be in the class $S(A,B;C,D)$, $-1 \le D < C \le 1$, $-1 \le B < A \le 1$ if there exists $g \in S^*(A,B)$ such that $\frac{f(z)}{g(z)} \in P(C,D)$.

The following observations are obvious:

(i) $S(1,-1;1,-1) \equiv S$.

(ii) $S(A,B;1,-1) \equiv S(A,B)$.

In the present paper, we study the class $S(A,B;C,D)$ and obtain coefficient bounds, distortion theorem, argument theorem and some invariance properties.

2. Coefficient Bounds

Unless otherwise mentioned in the sequel, the only restrictions on the real constants A, B, C and D are that $-1 \le B < A \le 1$ and $-1 \le D < C \le 1$.

Lemma A[4] Let

$$p(z) = \frac{1+Az}{1+Bz} = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

then $$|p_n| \le (A-B), \; n \ge 1.$$

The bounds are sharp, being attained for the functions

$$P(z) = \frac{1+A\delta z^n}{1+B\delta z^n}, |\delta| = 1.$$

Lemma B[1] If $g(z)=z+\sum_{k=2}^{\infty} b_k z^k \in S^*(A,B)$, then

$$|b_2| \le (A-B).$$

$$|b_n| \le \frac{1}{(n-1)!}\prod_{j=2}^{n}\left(A-(j-1)B\right),\ A-(n-1)B \ge (n-2),\ n\ge 3.$$

Theorem 1. If $f(z)\in A$, then for $A-(n-1)B\ge(n-2),\ n\ge 2$,

$$|a_n| \le \frac{1}{(n-1)!}\prod_{j=2}^{n}\left(A-(j-1)B\right)+(C-D)\left[1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\left(A-(j-1)B\right)\right]. \tag{2.1}$$

The bounds are sharp.

Proof. Since $f \in S(A,B;C,D)$, there exists a function $g(z)=z+\sum_{k=2}^{\infty} b_k z^k \in S^*(A,B)$ and a Schwarz function $w(z)$ such that

$$\frac{f(z)}{g(z)}=\frac{1+Cw(z)}{1+Dw(z)}=1+\sum_{k=1}^{\infty} p_k z^k\ ,\ \ z\in E. \tag{2.2}$$

From (2.2), it yields

$$z+\sum_{k=2}^{\infty} a_k z^k=\left(z+\sum_{k=2}^{\infty} b_k z^k\right)\left(1+\sum_{k=1}^{\infty} p_k z^k\right). \tag{2.3}$$

On equating the coefficients of z^n in (2.3), we have

$$a_n=b_n+p_1b_{n-1}+p_2b_{n-2}+.....+p_{n-1}. \tag{2.4}$$

Using Lemma A, (2.4) becomes

$$|a_n|\le|b_n|+(C-D)\left[|b_{n-1}|+|b_{n-2}|+.....+|b_2|+1\right]. \tag{2.5}$$

(2.5) together with Lemma B, yields

$$|a_n| \le \frac{1}{(n-1)!}\prod_{j=2}^{n}\left(A-(j-1)B\right)$$
$$+(C-D)\left[\frac{1}{(n-2)!}\prod_{j=2}^{n-1}\left(A-(j-1)B\right)+\frac{1}{(n-3)!}\prod_{j=2}^{n-2}\left(A-(j-1)B\right)+...+(A-B)+1\right]$$

which on simplification, yields (2.1). That completes the proof.

The function $f_0(z)$ defined by,

$$f_0(\mathrm{z})=\begin{cases}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)\mathrm{z}(1+\mathrm{B}\delta_2 z)^{\frac{(\mathrm{A-B})}{B}};\mathrm{B}\neq 0\\ \left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)z\exp(A\delta_2 z); B=0, |\delta_1|=|\delta_2|=1,\end{cases} \tag{2.6}$$

shows that the bound (2.1) is sharp for each $n\geq 2$.

For $A=C=1$ and $B=D=-1$, Theorem 1 agrees with the following result of Reade [7].

Corollary. If f is a close-to-star function, then $|a_n|\leq n^2$, $n\geq 2$.

3. Distortion Theorem

Lemma C[5] If $g\in S^*(A,B)$, then for $|z|=r<1$,

$$r(1-Br)^{\frac{(A-B)}{B}}\leq|g(z)|\leq r(1+Br)^{\frac{(A-B)}{B}}, B\neq 0;$$

$$r\exp(-Ar)\leq|g(z)|\leq r\exp(Ar), B=0.$$

Theorem 2. For $f\in S(A,B;C,D)$ and for $|z|=r$, $0<r<1$, we have

$$r\left(\frac{1-Cr}{1-Dr}\right)(1-Br)^{\frac{(A-B)}{B}}\leq|f(z)|\leq r\left(\frac{1+Cr}{1+Dr}\right)(1+Br)^{\frac{(A-B)}{B}}, B\neq 0; \tag{3.1}$$

$$r\left(\frac{1-Cr}{1-Dr}\right)\exp(-Ar)\leq|f(z)|\leq r\left(\frac{1+Cr}{1+Dr}\right)\exp(Ar), B=0. \tag{3.2}$$

Estimates are sharp.

Proof. From (2.2), we have

$$|f(z)|=|g(z)|\left|\frac{1+Cw(z)}{1+Dw(z)}\right|.$$

It is easy to show that

$$\frac{1-Cr}{1-Dr}\leq\left|\frac{1+Cw(z)}{1+Dw(z)}\right|\leq\frac{1+Cr}{1+Dr}. \tag{3.3}$$

Since $g\in S^*(A,B)$, thus due to Lemma C, the result is obvious.

Equality signs in (3.1) and (3.2) are attained by the function defined by (2.6).

4. Argument Theorem

Lemma D[3]. If $g\in S^*(A,B)$, then for $|z|=r<1$,

$$\left|\arg\frac{g(z)}{z}\right| \le \frac{(A-B)}{B}\sin^{-1}(Br), B \ne 0;$$

$$\left|\arg\frac{g(z)}{z}\right| \le Ar, B = 0.$$

Theorem 3. For $f \in S(A,B;C,D)$,

$$\left|\arg\frac{f(z)}{z}\right| \le \frac{(A-B)}{B}\sin^{-1}(Br) + \sin^{-1}\frac{(C-D)r}{1-CDr^2}, B \ne 0; \tag{4.1}$$

$$\left|\arg\frac{f(z)}{z}\right| \le Ar + \sin^{-1}\frac{(C-D)r}{1-CDr^2}, B = 0. \tag{4.2}$$

These inequalities are sharp.

Proof. It is easy to show that, the transformation $\frac{f(z)}{g(z)} = \frac{1+Cw(z)}{1+Dw(z)}$ maps $|w(z)| \le r$ onto the circle

$$\left|\frac{f(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2}\right| \le \frac{(C-D)r}{(1-D^2r^2)}, \ |z| = r.$$

Therefore

$$\left|\arg\frac{f(z)}{g(z)}\right| \le \sin^{-1}\frac{(C-D)r}{1-CDr^2}. \tag{4.3}$$

Since

$$\left|\arg\frac{f(z)}{z}\right| \le \left|\arg\frac{f(z)}{g(z)}\right| + \left|\arg\frac{g(z)}{z}\right|. \tag{4.4}$$

Using (4.3) and Lemma D in (4.4), we obtain (4.1) and (4.2). Equality signs in (4.1) and (4.2) hold for the functions $f_1(z)$ and $f_2(z)$, respectively, where

$$f_1(z) = \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right) z(1+B\delta_2 z)^{\frac{(A-B)}{B}},$$

$$f_2(z) = \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right) z\exp(A\delta_2 z),$$

and $$\delta_1 = \frac{ir}{z}, \quad \delta_2 = \frac{r}{z}\left[-Br + i\left(1-B^2r^2\right)^{1/2}\right].$$

5. Invariance Properties

The Convolution or **Hadamard Product** of two functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$

is denoted by $f * g$ and defined as $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$.

Lemma E[9]. Let ϕ be convex and g be starlike in E. Then for F analytic in E with $F(0)=1$,

$\frac{\phi * Fg}{\phi * g}(E)$ is contained in the convex hull of $F(E)$.

Lemma F[10]. If $g \in S^*(A, B)$, then for ϕ convex , $\phi * g \in S^*(A,B)$.

Theorem 4. If ϕ is convex and $f \in S(A,B;C,D)$, then $\phi * f \in S(A,B;C,D)$.

Proof. For $f \in S(A,B;C,D)$, there exists $g \in S^*(A,B)$ and $F \in P(C,D)$ such that

$$f(z) = g(z) F(z).$$

Since $\frac{1+Cz}{1+Dz}$ is convex in E, by Lemma E

$$\frac{\phi * f}{\phi * g} = \frac{\phi * Fg}{\phi * g} \prec \frac{1+Cz}{1+Dz} \quad (z \in E) \tag{5.1}$$

ϕ being a convex function.

From Lemma F, $\phi * g \in S^*(A,B)$. Thus (5.1) is equivalent to $\phi * f \in S(A,B;C,D)$.

Corollary. If $f \in S(A,B;C,D)$, then so are

(i) $F_1(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \operatorname{Re}\gamma > 0$

and

(ii) $F_2(z) = \int_0^z \frac{f(t) - f(st)}{t - st} dt, |s| \le 1, s \ne 1.$

Proof. Observe that $F_j(z) = (h_j * f), j = 1,2,$ where

$$h_1(z) = \sum_{k=1}^{\infty} \frac{1+\gamma}{k+\gamma} z^k, \operatorname{Re}\gamma > 0 \text{ and } h_2(z) = \sum_{k=1}^{\infty} \frac{1-s^k}{(1-s)k} z^k = \frac{1}{1-s} \log\left(\frac{1-sz}{1-z}\right), |s| \le 1, s \ne 1.$$

Since due to [4], h_1 is proved to be convex ,thus h_2 also is convex.

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