Some Study on Uncertainty Principle

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Abstract: In the present paper, we introduce a new form of uncertainty principle in terms of moment generating function (m.g.f) and find the upper bound of the product for the time spread $\sigma_t$ and frequency spread $\sigma_\omega$ of a signal $f(t)\in L^2(R)$ in terms of the Mathematical expectation and also in case of self-adjoint operators. Further, we find the estimation of frequency at any instant of time and vice-versa.

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1. Introduction

The Uncertainty Principle (UP) roughly speaking states that a non-zero function and its Fourier transform cannot be both sharply localised. The classical uncertainty principle was established by Heisenberg [4] bringing fundamental problems in quantum mechanics to the point. The position and momentum of particles cannot be both determined explicitly but only in parabolistic sense with a certain “uncertainty”. The mathematical equivalent is that a vector in a Hilbert space and its Fourier transform cannot be arbitrary localised. This is the fundamental problem in time-frequency analysis, where one would like to have basis of vectors well localised in both time and frequency. Recently, a number of papers have been published including uncertainty principles for periodic functions [5,6,9], functions on
intervals [8], on sphere [3,5]. These results provide qualitative and quantitative tools in order to determine time-frequency localisation of basis functions e.g Wavelets in different function spaces.

The uncertainty principle states that, if $\sigma_t$ and $\sigma_\omega$ are the time and frequency spread of a signal $f(t) \in L^2(R)$ with (i) $\int_{-\infty}^{\infty} |f(t)|^2 dt = 1$, (ii) $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = 1$, (By Plancheral Theorem) where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$. Then

$$\sigma_t \sigma_\omega \geq \frac{1}{2} \quad (1.1)$$

where, $\sigma_t$ and $\sigma_\omega$ are the duration and bandwidth of a signal $f(t) \in L^2(R)$ defined by

$$\sigma_t^2 = \int_{-\infty}^{\infty} (t - t_0)^2 |f(t)|^2 dt, \quad (1.2)$$

and

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} (\omega - \omega_0)^2 |\hat{f}(\omega)|^2 d\omega, \quad (1.3)$$

$t_0$ and $\omega_0$ are the means of time $t$ and frequency $\omega$ respectively and are defined by

$$t_0 = \int_{-\infty}^{\infty} t |f(t)|^2 dt, \quad (1.4)$$

and

$$\omega_0 = \int_{-\infty}^{\infty} \omega |\hat{f}(\omega)|^2 d\omega, \quad (1.5)$$

where, $\hat{f}(\omega)$ is the Fourier transform of $f(t)$. The Fourier transform of $f(t) \in L^1(R)$ is defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-it\omega)f(t)dt. \quad (1.6)$$

If $\hat{f}(\omega)$ is also in $L^1(R)$, then the inversion formula holds, that is,
\[ f(t) = F^{-1}(f(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(it\omega)\hat{f}(\omega)d\omega, \] (1.7)

and by Plancheral Theorem, \( \|\hat{f}(\omega)\|^2 = \|f(t)\|^2 \). The equality in (1.1) holds if \( f(t) \) is a Gaussian signal given by

\[ f(t) = c\exp(-bt^2), b > 0, c \in \mathbb{R}. \]

**Definition 1.10:** The moment generating function (m.g.f) of a random variable \( X \) (about origin) having the density function \( f(x) \) is given by

\[ M_X(t) = \int_{-\infty}^{\infty} \exp(tx)f(x)dx, \quad \text{(For continuous distribution)} \] (1.8)

and

\[ M_X(t) = \sum_{-\infty}^{\infty} \exp(tx)f(x). \quad \text{(For discrete distribution).} \] (1.9)

The integration or summation being extended to the entire range of \( X \). Here \( t \) being the real parameter and is being assumed that the right sides of (1.8) and (1.9) are absolutely convergent for some positive number \( h \) such that \(-h < t < h\).

**Definition 1.11:** For a random variable \( X \), the variance of \( X \) is defined as

\[ \sigma_X^2 = M_X''(0) - [M_X'(0)]^2, \]

where \( M_X'(0) \) and \( M_X''(0) \) are the first and second derivatives of \( M_X(t) \) at \( t=0 \).

In the present paper, we derive the alternative form of the Heisenberg’s inequality in terms of the moment generating function (m.g.f). We also find the upper bound for the product \( \sigma_t \sigma_\omega \) of a signal \( f(t) \in L^2(\mathbb{R}) \) and also in case of self-adjoint operators.

**2. Main Results**

**Theorem 2.1.** Let \( \sigma_t \) and \( \sigma_\omega \) denote the time and frequency spread of a signal \( f(t) \in L^2(\mathbb{R}) \) with \( \|f(t)\| = 1 \) and \( m(t) \) is the moment generating function, then

\[ A_{t,\omega} - B_{t,\omega} \geq \frac{1}{4}, \]
where,

\[ A_{t,\omega} = M''_t(0)M''_\omega(0) + [M'_t(0)M'_\omega(0)]^2 \]

and

\[ B_{t,\omega} = M''_t(0)[M''_\omega(0)]^2 + [M'_t(0)]^2M''_\omega(0). \]

**Proof.** For a given signal \( f(t) \in L^2(\mathbb{R}) \) with \( ||f(t)|| = 1 \), Clearly, \( |f(t)|^2 \) is a density function of the signal \( f(t) \) in time domain \( t \) because, \( \int_{-\infty}^{\infty} |f(t)|^2 = ||f(t)|| = 1 \) and since by Plancherel Theorem, We have

\[ \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega. \]

Therefore, \( |\hat{f}(\omega)|^2 \) is the density function for the signal \( \hat{f}(\omega) \) in frequency domain \( \omega \). So, for a given signal \( f(t) \in L^2(\mathbb{R}) \) with \( ||f(t)|| = 1 \), the moment generating functions for the time \( t \) and frequency \( \omega \) are given by

\[ M_t(h) = \int_{-\infty}^{\infty} e^{th}|f(t)|^2 dt, \tag{2.10} \]

and

\[ M_\omega(h) = \int_{-\infty}^{\infty} e^{\omega h}|\hat{f}(\omega)|^2 d\omega, \tag{2.11} \]

\( \hat{f}(\omega) \) being the Fourier transform of \( f(t) \).

Since for a signal \( f(t) \) with \( ||f(t)|| = 1 \), we have

\[ \sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}. \tag{2.12} \]

Since

\[ \sigma_t^2 = M''_t(0) - [M'_t(0)]^2 \]

and

\[ \sigma_\omega^2 = M''_\omega(0) - [M'_\omega(0)]^2 \]

Equation (2.12) gives,

\[ \{M''_t(0) - [M'_t(0)]^2\} \{M''_\omega(0) - [M'_\omega(0)]^2\} \geq \frac{1}{4}. \]
i.e,

\[ M''(0)M''(0) + [M'(0)]^2[M''(0)]^2 - \{M''(0)[M'(0)]^2 + [M'(0)]^2M''(0)\} \geq \frac{1}{4}. \] (2.13)

Let,

\[ A_{t,\omega} = M''(0)M''(0) + [M'(0)]^2[M''(0)], \] \hspace{1cm} (2.14)

and

\[ B_{t,\omega} = M''(0)[M'(0)]^2 + [M'(0)]^2M''(0); \] \hspace{1cm} (2.15)

and substitute (2.14) and (2.15) in (2.13), we obtain

\[ A_{t,\omega} - B_{t,\omega} \geq \frac{1}{4}. \] \hspace{1cm} (2.16)

Equation (2.16) gives the Heisenberg’s inequality in terms of the moment generating function (m.g.f)

**Remark 2.2** It can be seen that the equality holds good in (2.16), if \( f(t) \) is a Gaussian signal i.e; \( f(t) = ce^{-bt^2}, b > 0, c \in R \).

We know that, the lower bound for the product \( \sigma_t\sigma_\omega \) exist, but the question is, what about the upper bound? The answer is given in the following result.

**Definition 2.3.** Suppose \( f(x) \) is the density function of a random variable \( X \), then the mathematical expectation of \( \phi(x) \) is given by

\[ E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)f(x)dx \] \hspace{1cm} (2.17)

similarly, for a function \( \phi(x, y) \) in two variables, the mathematical expectation is given by

\[ E[\phi(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, y)f(x, y)dxdy, \] \hspace{1cm} (2.18)

where \( f(x, y) \) is the joint density function of the random variables \( X \) and \( Y \).

**Theorem 2.4** If \( f(t) \in L^2(R) \) is a signal with \( ||f|| = 1 \), \( \sigma_t \) and \( \sigma_\omega \) being the time and frequency spread of \( f(t) \), then

\[ \sigma_t\sigma_\omega \leq t_0\omega_0 - E(t\omega), \]

where \( t_0 \) and \( \omega_0 \) are the central values of \( f(t) \) about time \( t \) and frequency \( \omega \).
Proof. The correlation coefficient of time \( t \) and frequency \( \omega \) regarded as random variables of a signal \( f(t) \in L^2(R) \) is given by

\[
\rho(t, \omega) = \frac{\text{cov}(t, \omega)}{\sigma_t \sigma_\omega},
\]

or

\[
\rho(t, \omega) = \frac{E(t \omega) - t_0 \omega_0}{\sigma_t \sigma_\omega},
\]

(2.19)

where \( t_0 \) and \( \omega_0 \) are defined in equations (1.4) and (1.5) respectively. Since the correlation coefficient lies between -1 and +1, therefore we have,

\[-1 \leq \rho(t, \omega) \leq +1,
\]

thus using equation (2.19), we have,

\[-1 \leq \frac{E(t \omega) - t_0 \omega_0}{\sigma_t \sigma_\omega} \leq +1.
\]

Now taking the left hand inequality, we get

\[
\frac{E(t \omega) - t_0 \omega_0}{\sigma_t \sigma_\omega} \geq -1,
\]

which implies,

\[
\sigma_t \sigma_\omega \leq t_0 \omega_0 - E(t \omega)
\]

(2.20)

where,

\[
E(t \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t \omega \phi(t, \omega) dt \, d\omega,
\]

and has been defined in (2.18), \( \phi(t, \omega) \) being the joint density function of \( t \) and \( \omega \) when regarded as random variables. The inequality (2.20) thus gives the upper bound of the product \( \sigma_t \sigma_\omega \).

3. Uncertainty Principle for Self-adjoint Operators

Let \( H \) be a Hilbert space with inner product \( \langle ., . \rangle \) and with the norm \( ||.|| = \langle ., . \rangle^{\frac{1}{2}} \). Let \( A, B \) be the two self-adjoint operators on \( H \), with domains \( D(A) \) and \( D(B) \). Then the domains of the product \( AB \) and \( BA \) are

\[
D(AB) = \{ f \in D(B) : Bf \in D(A) \},
\]

and

\[
D(BA) = \{ f \in D(A) : Bf \in D(B) \}.
\]
The commutator and anti-commutator are, respectively, defined as

\[
[A, B] = AB - BA \quad \text{on} \quad D([A, B]) = D(AB) \cap D(BA),
\]

and

\[
[A, B]_+ = AB + BA \quad \text{on} \quad D([A, B]) = D(AB) \cap D(BA).
\]

**Definition 3.1.** Let \( f(t) \in L^2(R) \) and A, B be two self-adjoint operators on \( L^2 \), with domains \( D(A) \) and \( D(B) \). Then the mean of the operator is defined by

\[
\langle A \rangle_f = \langle Af, f \rangle = \int Af \, dt,
\]

and the variance is

\[
\sigma^2_A(f) = \langle (A - \langle A \rangle_f I)^2 \rangle = \int f(A - \langle A \rangle_f I)^2 f \, dt,
\]

and the covariance of operators A and B is defined by

\[
Cov_{A,B}(f) = \frac{1}{2} \langle AB + BA \rangle_f - \langle A \rangle_f \langle B \rangle_f
\]

\[
= \frac{1}{2} \langle [A, B]_+ \rangle_f - \langle A \rangle_f \langle B \rangle_f
\]

\[
= \frac{1}{2} \langle [A - \langle A \rangle_f I, B - \langle B \rangle_f I]_+ \rangle_f,
\]

where, I is the identity operator.

**Lemma 3.2.** (See[1]). Let \( f(t) \in L^2(R) \) and B be a self-adjoint operator on \( L^2(R) \) with domain \( D(B) \). Then

\[
\sigma^2_f(B) = \int \left[ \text{Im} \left( \frac{Bf}{f} \right) \right]^2 |f(t)|^2 dt + \int \left[ \text{Re} \left( \frac{Bf}{f} \right) - \langle B \rangle_f \right]^2 |f(t)|^2 dt,
\]

where \( \text{Im} \left( \frac{Bf}{f} \right) \) and \( \text{Re} \left( \frac{Bf}{f} \right) \) denote, respectively, the imaginary part and real part of \( \frac{Bf}{f} \).

**Proof.**

\[
\sigma^2_f(B) = \int f(t)(B - \langle B \rangle_f)^2 f(t) \, dt
\]

\[
= \int |(B - \langle B \rangle_f) f(t)|^2 \, dt
\]

\[
= \int |Bf(t) - \langle B \rangle_f f(t)|^2 \, dt
\]
\[
\left( \int \left| \left( \frac{B f(t)}{f(t)} - \langle B \rangle_f \right) f(t) \right|^2 dt \right)^{-1} = \int \left| \left[ \left( \frac{B f(t)}{f(t)} - \langle B \rangle_f \right) f(t) \right|^2 dt \right| \int \left| \left( \frac{R e \left( \frac{B f(t)}{f(t)} - \langle B \rangle_f \right) f(t) \right|^2 dt \right|
\]

**Theorem 3.3.** (See [2].) Let A, B be two self-adjoint operators on \(L^2(R)\), with domains D(A) and D(B). Assume that \(f \in D(AB) \cap D(BA)\) and \((Af)(Bf) = fABf\). Then

\[
\sigma_A^2(f) \sigma_B^2(f) \geq \frac{1}{4} |\langle [A, B] \rangle_f| + \left[ \int |A f(i) - \langle A \rangle_f f(i) | \left| \left[ \frac{B f}{f} \right] - \langle B \rangle_f \right| |f(i)| dt \right]^2.
\]

The equality is attained if and only if there exists positive numbers \(\zeta, \epsilon\) such that

\[
|\langle A - \langle A \rangle_f \rangle f| = \zeta \left| Im \left( \frac{B f}{f} \right) f \right| = \epsilon \left| Re \left( \frac{B f}{f} \right) - \langle B \rangle_f \right| f |. \tag{3.13}
\]

Since the theorem (3.3) gives the lower bound for the product \(\sigma_A^2(f) \sigma_B^2(f)\) for the self-adjoint operators A and B together with the condition defined in the theorem (3.2). Our next theorem gives the upper bound for the same.

**Theorem 3.4.** Let \(f(t) \in L^2(R)\) and A, B be two self-adjoint operators on \(L^2(R)\), with domains D(A) and D(B). Then

\[
\sigma_A(f) \sigma_B(f) \leq \frac{1}{2} \left\langle \left[ \langle A \rangle_f I - A, B - \langle B \rangle_f I \right]_+ \right\rangle.
\]

**Proof.** The correlation-coefficient for the self adjoint-operators A, B of a signal \(f(t) \in L^2(R)\) is given by

\[
\tau_{A,B}(f) = \frac{Cov_{A,B}(f)}{\sigma_A(f) \sigma_B(f)} = \frac{1}{2} \left\langle \left[ A - \langle A \rangle_f I, B - \langle B \rangle_f I \right]_+ \right\rangle \frac{\sigma_A(f) \sigma_B(f)}{\sigma_A(f) \sigma_B(f)},
\]

where, I is the identity operator.

Since,

\[-1 \leq \tau_{A,B}(f) \leq +1,
\]

implies

\[-1 \leq \frac{1}{2} \left\langle \left[ A - \langle A \rangle_f I, B - \langle B \rangle_f I \right]_+ \right\rangle \leq +1,
\]

or
\[ \sigma_A(f)\sigma_B(f) \leq \left[ \frac{1}{2} \langle [A > f I - A, B - < B > f I]_+ \rangle \right], \]  
\( \tag{3.15} \)

Equation (3.15) gives the upper bound for the product \( \sigma_A(f)\sigma_B(f) \) for self-adjoint operators.

4. Estimation of Time and Frequency and Deviation of Estimated Values

For any signal \( f(t) \in L^2(R), (t, \omega) \) are a continuous set of points on the plane, so given at any instant of time \( t \), it is required to find the estimated value of frequency \( \omega \) of a signal \( f(t) \) and vice versa. The requirement is to find a curve which best estimates our requirements. For the set of \( (t, \omega) \) points, regarding \( t, \omega \) as random variables, the line of regression of frequency \( \omega \) on time \( t \) is given by

\[ \omega - \omega_0 = r \frac{\sigma_\omega}{\sigma_t} (t - t_0), \]
\( \tag{4.11} \)

where \( r \frac{\sigma_\omega}{\sigma_t} \) is the slope of the line of regression of frequency \( \omega \) on time \( t \). Thus the estimated value of frequency \( \omega \) is

\[ \hat{\omega} = \omega_0 + r \frac{\sigma_\omega}{\sigma_t} (t - t_0). \]
\( \tag{4.12} \)

The above equation gives the estimated value of frequency \( \omega \) when time \( t \) is known but is not reversible. The equation (4.12) after introducing the inequality

\[ \sigma_t \sigma_\omega \geq \frac{1}{2} \]
\( \tag{4.13} \)

of a signal \( f(t) \) which is square integrable becomes

\[ \hat{\omega} \geq \omega_0 + \frac{r}{2\sigma_t^2} (t - t_0). \]
\( \tag{4.14} \)

and

\[ \hat{\omega} \leq \omega_0 + 2r \sigma_\omega^2 (t - t_0). \]
\( \tag{4.15} \)

Thus the estimated value of frequency \( \omega \) for any instant of time lies between the bounds \( \omega_0 + \frac{r}{2\sigma_t^2} (t - t_0) \) and \( \omega_0 + 2r \sigma_\omega^2 (t - t_0) \).

Adopting a similar procedure, the bounds for the estimated value \( \hat{t} \) for the time \( t \) for any instant of frequency are \( t_0 + \frac{r}{2\sigma_\omega^2} (\omega - \omega_0) \) and \( t_0 + 2r \sigma_\omega^2 (\omega - \omega_0) \).

Next we find the deviation of the estimated value from the actual one. The deviation of estimated \( \hat{\omega} \) of frequency \( \omega \) from its actual value is best given by

\[ S^2_\omega = E(\omega - \hat{\omega})^2 \] called the residual variance. Thus
\[ S_\omega^2 = E\left[\omega - \omega_0 - \frac{\sigma_\omega}{\sigma_t} (t - t_0)\right]^2, \]

or

\[ S_\omega^2 = \sigma_\omega^2 E\left[\left(\frac{\omega - \omega_0}{\sigma_\omega}\right) - r\left(\frac{t - t_0}{\sigma_t}\right)\right]^2. \]  

(4.16)

Let \( \omega' = \frac{\omega - \omega_0}{\sigma_\omega} \) and \( t' = \frac{t - t_0}{\sigma_t} \) be two standardized random variables. Therefore equation (4.16) reduces to

\[ S_\omega^2 = \sigma_\omega^2 E\left[\omega'^2 - 2r\omega't' + rt'^2\right]. \]  

(4.17)

Clearly, \( E(\omega'^2) = E(t'^2) = 1 \) and \( E(\omega't') = r \), therefore equation (4.17) becomes

\[ S_\omega^2 = \sigma_\omega^2 (1 - r^2) \]

or

\[ S_\omega = \sigma_\omega (1 - r^2)^{\frac{1}{2}} \]  

(4.18)

Thus the standard error estimate of frequency is given by equation (4.18). Similarly, the standard error estimate of time \( t \) is given by

\[ S_t = \sigma_t (1 - r^2)^{\frac{1}{2}} \]  

(4.19)

If \( r = \pm 1 \) both \( S_\omega \) and \( S_t = 0 \), so that each deviation is zero and the two lines of regression are coincident (they cannot be parallel because both have a point of intersection \( (t_0, \omega_0) \)).

**Theorem 4.1** For a signal \( f(t) \in L^2(R) \), if \( S_t \) and \( S_\omega \) denote the standard error estimates of time \( t \) and frequency of the signal \( f(t) \), then

\[ S_t S_\omega \geq \frac{1}{2} (1 - r^2) \]  

(4.20)

**Proof.** Since for a signal \( f(t) \in L^2(R) \), we have

\[ \sigma_t \sigma_\omega \geq \frac{1}{2} \]

Combining the above inequality and equations (4.18) and (4.19), we obtain (4.20).

**Theorem 4.2** For a signal \( f(t) \in L^2(R) \), then \( r(\omega, \hat{\omega}) = r(t, \omega) \).

**Proof.** Since

\[ r(\omega, \hat{\omega}) = \frac{\text{Cov} (\hat{\omega}, \omega)}{\sigma_\omega \sigma_\hat{\omega}}, \]  

(4.21)

we have,
Remark 4.3 Theorem 4.2 shows that the correlation coefficient between the observed and estimated value of frequency $\omega$ is same as the correlation coefficient between time $t$ and frequency $\omega$. Similarly, the correlation coefficient between the observed and estimated value of time $t$ is same as the correlation coefficient between time $t$ and frequency $\omega$.

References


