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Article

Chebyshev Wavelets Method for Delay Differential Equations

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Abstract: Chebyshev Wavelets Method (CWM) is applied to find numerical solutions of Delay Differential Equations. Computational work is fully supportive of compatibility of proposed algorithm and hence the same may be extended to other physical problems also. A very high level of accuracy explicitly reflects the reliability of this scheme for such problems.

Keywords: Chebyshev Wavelets Method, Delay Differential Equations, exact solutions.

Mathematics Subject Classification: 35Q79, 42C15, 39B9.

1. Introduction

Wavelet theory [2, 3, 5, 9-13, 15] is one of the relatively new technique which is being utilized for solving wide range of physical problems related to various branches of engineering and applied sciences. With the passage of time, lot of rapid developments is taking place which are helpful in increasing the accuracy of this scheme. The most common related schemes are Haar Wavelets [11], Harmonic Wavelets of successive approximation [3], CAS Wavelets [15], Legendre Wavelets [9-10, 12-13] and Chebyshev Wavelets [2, 5]. In the similar context, we merge Chebyshev polynomials with the traditional wavelet technique. The modified version which is called Chebyshev Wavelets Method (CWM) proves to be fully compatible with the complexity of the given problems and obtained results are extremely accurate. In particular, we apply Chebyshev Wavelets Method (CWM) on Delay

Differential Equations. It is worth mentioning that such problems arise in modeling biological systems such as gestation and maturation see [1, 4-8, 14]. It is observed that CWM is very user friendly but is extremely accurate. The error estimates explicitly reveal the very high accuracy level of the suggested technique.

2. Properties of Chebyshev Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [12] $\psi_{a,b}(x)=|a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right),\ a,\ b\in R,\ a\neq 0.$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k},\ b=nb_0a_0^{-k},\ a_0>1,\ b_0>0,$ we have the following family of discrete wavelets

$\psi_{k,n}(x)=|a|^{\frac{k}{2}}\psi\left(a_0^k x-nb_0\right), k,n\in\mathbb{Z},$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$, then $\psi_{k,n}(x)$ form an orthonormal basis.

The chebyshev wavelets $\psi_{n,m}(x)=\psi(k,n,m,x)$ involve four arguments $n=1,2,\cdots,2^{k-1}$, k is assumed any positive integer, m is the degree of the second chebyshev polynomials and it is the normalized time. They are defined on the interval $[0,1)$ as

$$\psi_{n,m}(x)=\begin{cases}2^{\frac{k}{2}}\tilde{T}_m\left(2^k x-2n+1\right), & \frac{n-1}{2^{k-1}}\leq x<\frac{n}{2^{k-1}},\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

where $\tilde{T}_m(x)=\sqrt{\frac{2}{\pi}}T_m(x),$ (2)

$m=0,1,2,\cdots,M-1$. In eq. (2) the coefficients are used for orthonormality. Here $T_m(x)$ are the second Chebyshev polynomials of degree m with respect to the weight function $w(x)=\sqrt{1-x^2}$ on the interval $[-1,1]$, and satisfy the following recursive formula

$T_0(x)=1,\ \ T_1(x)=2x,$

$T_{m+1}(x)=2xT_m(x)-T_{m-1}(x),\ \ m=1,2,3,\cdots.$

Chebyshev Wavelet Method (CWM): In the present paper, we consider the Delay Differential Equation of the form $y'''(x) = f(y) + g(x)\,y\left(\frac{x}{a}\right),\ 0 < x < b,$ (3)

with boundary condition $y(0) = \alpha_0,\ y'(0) = \alpha_1, y''(0) = \alpha_2,$

where $g(x)$ is a source term function, $f(y)$ is a given continuous linear or nonlinear function and $\alpha_i,\ i = 0,1,2$ are real finite constants.

The solution of the equation (3) can be expanded as a Chebyshev wavelets series as follows:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x),$$

where $\psi_{n,m}(x)$ is given by the equation (1). We approximate $y(x)$ by the truncated series

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(x). \tag{4}$$

Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients $c_{10,}c_{11},\ldots,c_{1M-1},\ c_{20,}c_{21},\ldots,c_{2M-1},\ldots,c_{2^{k-1}0,}c_{2^{k-1}1},\cdots,c_{2^{k-1}M-1}$.

Since three conditions are furnished by the boundary conditions, namely

$$\begin{aligned} u_{k,M}(0) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(0) = \alpha_0, \\ \frac{d}{dx}u_{k,M}(0) &= \frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(0) = \alpha_1, \\ \frac{d^2}{dx^2}u_{k,M}(0) &= \frac{d^2}{dx^2}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(0) = \alpha_2, \end{aligned} \tag{5}$$

We see that there should be $2^{k-1}M-3$ extra conditions to recover the unknown coefficients c_{nm}. These conditions can be obtained by substituting equation (4) in equation (3);

$$\frac{d^3}{dx^3}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-4} c_{nm}\psi_{n,m}(x) = f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-4} c_{nm}\psi_{n,m}(x)\right) + g(x)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-4} c_{nm}\psi_{n,m}\left(\frac{x}{a}\right). \tag{6}$$

We, now assume equation (6) is exact at $2^{k-1}M-3$ points x_i as follows:

$$\frac{d^3}{dx^3}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-4} c_{nm}\psi_{n,m}(x_i) = f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-4} c_{nm}\psi_{n,m}(x_i)\right) + g(x_i)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-4} c_{nm}\psi_{n,m}\left(\frac{x_i}{a}\right). \tag{7}$$

The best choice of the x_i points are the zeros of the shifted chebyshev polynomials of degree $2^{k-1}M-3$ in the interval $[0,1]$ that is $x_i = \frac{s_i+1}{2}$, where $s_i = \cos\left(\frac{(2i-1)\pi}{2^{k-1}M-1}\right),\ i = 1,\ldots,2^{k-1}M-3.$

Combine equation (5) and (7) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknown coefficients, c_{nm} .

Same procedure is repeated for delay differential equations of first and second order also.

3. Solution Procedure

Example 1. Consider the Delay differential equation of the form

$$y'(x) = \frac{1}{2}e^{\frac{x}{2}}y\left(\frac{x}{2}\right) + \frac{1}{2}y(x),$$

subject to the boundary condition $y(0) = 1$.

The exact solution of the above system is $y(x) = e^x$.

Table 1 and Fig. 1 show the comparison of the absolute error between exact solution and approximate solution for $M = 10$ and $k = 1$ by Chebyshev wavelet method (CWM)

Table 1: Numerical results of Example 1

x	Exact Solution	Approximate Solution	Error in CWM
0.0	1.000000000000000	1.000000000000000	1.20000E-999
0.1	1.105170918075648	1.105170917613166	4.62482E-10
0.2	1.221402758160170	1.221402757619899	5.40271E-10
0.3	1.349858807576003	1.349858807526211	4.97921E-11
0.4	1.491824697641270	1.491824698252717	6.11446E-10
0.5	1.648721270700128	1.648721271339750	6.39622E-10
0.6	1.822118800390509	1.822118800026426	3.64083E-10
0.7	2.013752707470477	2.013752706111536	1.35894E-09
0.8	2.225540928492468	2.225540928394773	9.76945E-11
0.9	2.459603111156950	2.459603113305860	2.14891E-09
1.0	2.718281828459045	2.718281814056567	1.44025E-08

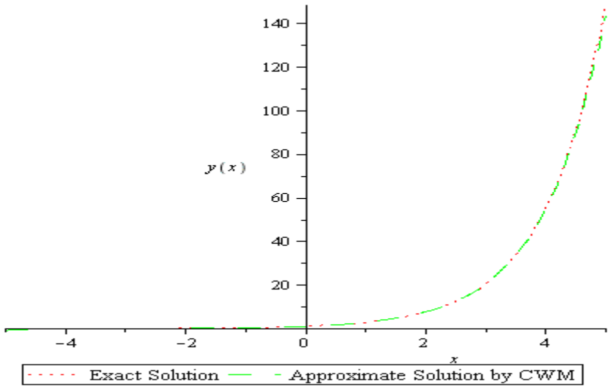

Fig. 1. Graphic comparison of exact and approximately numerical solutions of Example 1

Example 2. Consider the Delay differential equation of the form

$$y'(x) - y\left(\frac{x}{2}\right) = 0,$$

subject to the boundary condition $y(0) = 1$.

The exact solution of the above system is$y(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}k(k-1)}}{k!} x^k$.

Table 2 and Fig. 2 show the comparison of the absolute error between exact solution and approximate solution for $M = 10$ and $k = 1$ by Chebyshev wavelet method (CWM)

Table 2: Numerical results of Example 2

x	Exact Solution	Approximate Solution	Error in CWM
0.0	1.000000000000000	1.000000000000000	1.00000E-999
0.1	1.102520898518923	1.102520898518923	1.40533E-23
0.2	1.210167710940214	1.210167710940214	1.61101E-23
0.3	1.323067793243810	1.323067793243810	1.34194E-24
0.4	1.441350083507100	1.441350083507100	1.75775E-23
0.5	1.565145111746998	1.565145111746998	1.68744E-23
0.6	1.694585009792689	1.694585009792689	1.37309E-23
0.7	1.829803521189073	1.829803521189073	4.18853E-23
0.8	1.970936011130968	1.970936011130968	2.69883E-24
0.9	2.118119476428119	2.118119476428119	6.21111E-23
1.0	2.271492555501061	2.271492555501061	4.18581E-22

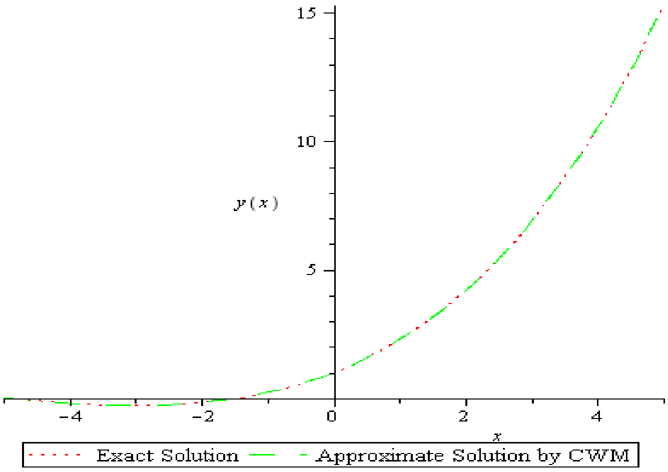

Fig. 2. Graphic comparison of exact and approximately numerical solutions of Example 2

Example 3. Consider the Delay differential equation of the form

$$y''(x) = \frac{3}{4}y(x) + y\left(\frac{x}{2}\right) + 2 - x^2,$$

subject to the boundary conditions $y(0) = 0, y'(0) = 0.$

The exact solution of the above system is $y(x) = x^2$.

Table 3 and Fig. 3 show the comparison of the absolute error between exact solution and approximate solution for $M = 10$ and $k = 1$ by Chebyshev wavelet method (CWM)

Table 3: Numerical results of Example 3

x	Exact Solution	Approximate Solution	Error in CWM
0.0	0.000000000000000	0.000000000000000	1.00000E-1000
0.1	0.010000000000000	0.010000000000000	2.50000E-1000
0.2	0.040000000000000	0.040000000000000	5.40000E-1000
0.3	0.090000000000000	0.090000000000000	6.30000E-1000
0.4	0.160000000000000	0.160000000000000	0.00000E+00
0.5	0.250000000000000	0.250000000000000	7.00000E-1000
0.6	0.360000000000000	0.360000000000000	1.00000E-999
0.7	0.490000000000000	0.490000000000000	3.00000E-1000
0.8	0.640000000000000	0.640000000000000	1.70000E-999
0.9	0.810000000000000	0.810000000000000	7.30000E-999
1.0	1.000000000000000	1.000000000000000	6.00000E-998

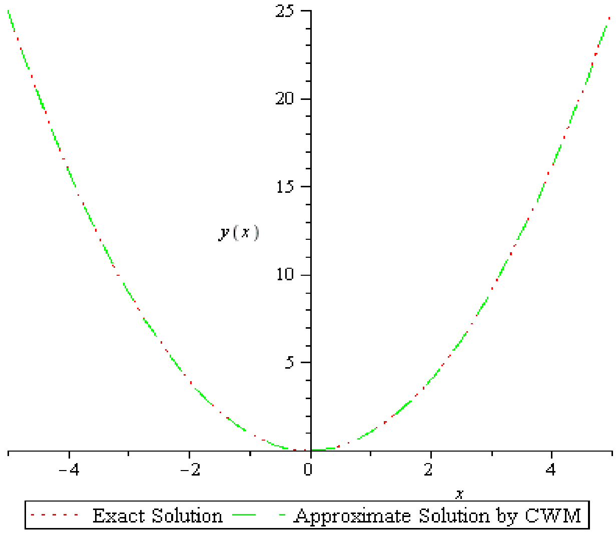

Fig. 3. Graphic comparison of exact and approximately numerical solutions of Example 3

Example 4. Consider the Delay differential equation of the form

$$y'''(x) + y(x) + y(x - 0.3) = \mathrm{e}^{-x+0.3},$$

subject to the boundary conditions

$y(0) = 1, y'(0) = -1, y''(0) = 1.$

The exact solution of the above system is $y(x) = e^{-x}$.

Table 4 and Fig. 4 show the comparison of the absolute error between exact solution and approximate solution for $M = 10$ and $k = 1$ by Chebyshev wavelet method (CWM)

Table 4: Numerical results of Example 4

x	Exact Solution	Approximate Solution	Error in CWM
0.0	1.000000000000000	1.000000000000000	0.00000E+00
0.1	0.904837418035960	0.904837417885435	1.50525E-10
0.2	0.818730753077982	0.818730751978017	1.09996E-09
0.3	0.740818220681718	0.740818217898862	2.78286E-09
0.4	0.670320046035639	0.670320042806103	3.22954E-09
0.5	0.606530659712633	0.606530660461651	7.49018E-10
0.6	0.548811636094026	0.548811643616776	7.52275E-09
0.7	0.496585303791410	0.496585296939500	6.85191E-09
0.8	0.449328964117222	0.449328837174443	1.26943E-07
0.9	0.406569659740599	0.406569089224090	5.70517E-07
1.0	0.367879441171442	0.367877627369487	1.81380E-06

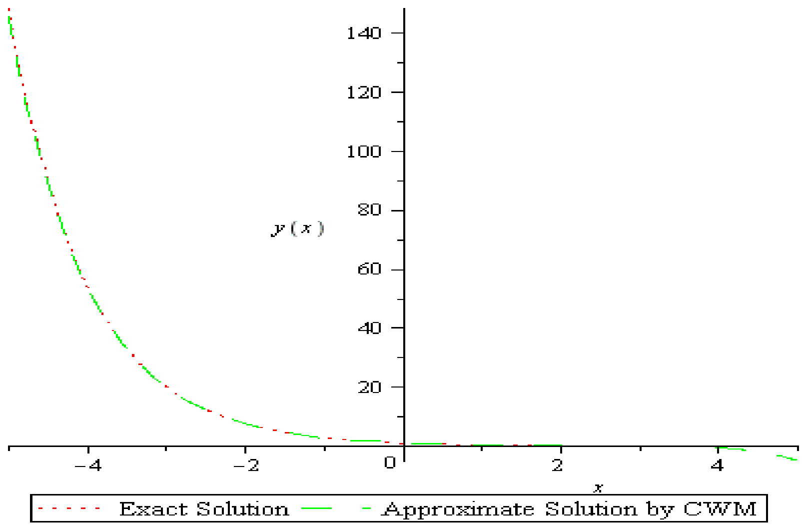

Fig. 4. Graphic comparison of exact and approximately numerical solutions of Example 4

4. Conclusion

Delay Differential Equations of different orders are successfully handled by using Chebyshev Wavelets Method (CWM). Computational work and numerical results explicitly reflect that CWM is very user-friendly but extremely accurate. It is also concluded that the same (CWM) may be extended to other linear and nonlinear diversified physical problems of complex nature.

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