Some New Sequence Spaces Derived from the Spaces of Bounded, Convergent and Null Sequences

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Abstract: In this article, we introduce the new paranormed sequence spaces $Z(p, A_{\lambda})$ consisting of all sequences whose generalized weighted mean $A_{\lambda}$ transforms are in the linear space $Z(p)$, where $Z(p)$ was defined by Maddox [Quart. J. Math. Oxford (2), 18(1967), 345–355] and $Z$ denotes one of the classical sequence spaces $\ell_\infty$, $c$ or $c_0$. Meanwhile, we have also presented the Schauder basis of $c_0(p, A_{\lambda})$ and $c(p, A_{\lambda})$ and computed its $\beta$- and $\gamma$-duals. In addition to this, the fact that sequence space $\{c_0\}_{A_{\lambda}}$ has AD property is shown and then the $f$-dual of the space $\{c_0\}_{A_{\lambda}}$ presented. In conclusion, we characterize the classes of matrix mappings from the sequence spaces $Z(p, A_{\lambda})$ to the sequence space $\mu$ and from the sequence space $\mu$ to the sequence spaces $Z(p, A_{\lambda})$.

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1. Preliminaries, Background and Notation

Many of the materials in the first chapter we discuss will be defined by means of infinite sequences or series. Therefore, most of this chapter is about the basic properties of them. Let us now consider some aspects of sequences. Let $X$ be a set. A sequence in $X$ is simply a function from $\mathbb{N}$ to $X$, i.e.,
\( \varphi : \mathbb{N} \to X \) is a sequence, we write also \((x_n)_{n \in \mathbb{N}}\) for \( \varphi \), where \( x_n := \varphi(n) \) is the \( n^{th} \) term of the sequence \( \varphi = (x_0, x_1, x_2, \ldots) \). Sequences in \( \mathbb{K} \) are called number sequences, and the \( \mathbb{K} \)-vector space \( \mathbb{K}^\mathbb{N} \) of all number sequences is denoted by \( \mathbb{K} \) or \( \mathbb{K}(\mathbb{K}) \). More precisely, one says \((x_n)\) is a real or complex sequence if \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \), where \( \mathbb{K} \) denotes either of fields \( \mathbb{R} \) and \( \mathbb{C} \). For \( m, n \in \mathbb{N} \), a function \( \varphi : m + \mathbb{N} \to X \) is also called a sequence in \( X \). That is, \((x_j)_{j \geq m} = (x_m, x_{m+1}, x_{m+2}, \ldots)\) is a sequence in \( X \) even though the indexing does not start with 0. This convention is justified, since after re-indexing using the function \( n \mapsto m + n \), the shifted sequence \((x_j)_{j \geq m}\) can be identified with the unusual sequence \((x_{m+k})_{k \in \mathbb{N}} \in \mathbb{X}^\mathbb{N} \). A sequence \((x_n)\) converges with limit \( a \) if each neighborhood of \( a \) contain almost all terms of the sequence. In this case we say that \((x_n)\) converges to \( a \) as \( n \) goes to \( \infty \). The set of all convergent sequences in \( \mathbb{K} \) we denote by \( c \). A sequence \((x_n)\) in \( \mathbb{K} \) is called a null sequence if it converges to zero. The set of all null sequences in \( \mathbb{K} \) we denote by \( c_0 \). A sequence is bounded if the set of its terms have an upper bound and a lower bound. The set of all bounded sequences is denoted by \( \ell_\infty \). Any vector subspace of \( \mathbb{K} = \mathbb{K}^\mathbb{N} \) is known as a sequence space. It is clear that the sets \( c \), \( c_0 \) and \( \ell_\infty \) equipped with a vector space structure, forms a sequence space. Also by \( b_1 \), \( c_1 \) and \( \ell_p \), we denote the spaces of all bounded, convergent, absolutely and \( p \)-absolutely convergent series, respectively.

We call a sequence space \( X \) with a linear topology a \( K \)-space if and only if each of the maps \( p_n : X \to \mathbb{R} \) defined by \( p_n(x) = x_n \) is continuous for all \( n \in \mathbb{N} \). A \( K \)-space \( X \) is called an \( FK \)-space if and only if \( X \) is a complete linear metric space. An \( FK \)-space whose topology is normal is called a \( BK \)-space, so a \( BK \)-space is a normed \( FK \)-space. The space \( \ell_p(1 \leq p < \infty) \) is a \( BK \)-space with \( ||x|| = (\sum_k |x_k|^p)^{1/p} \) and \( c_0 \) and \( \ell_\infty \) are \( BK \)-spaces with \( ||x|| = \sup_k |x_k| \). An \( FK \)-space \( X \) is said to have \( AK \) property, if \( \phi \subset X \) and \( \{e^k\} \) is a basis for \( X \), where \( e^k \) is a sequence whose only non-zero term is a 1 in \( k \)th place for each \( k \in \mathbb{N} \) and \( \phi = \text{span}\{e^k\} \), the set of all finitely non-zero sequences. If \( \phi \) is dense in \( X \), then \( X \) is an \( AL \)-space, thus \( AK \) implies \( AD \). For example, the spaces \( c_0 \), \( c_1 \) and \( \ell_p \) are \( AK \)-spaces, where \( 1 \leq p < \infty \).

The function \( g \) on \( X \) satisfies the properties of a paranorm

\[ g(\theta) = 0, \]
\[ g(x) = g(-x), \]
\[ g(x + y) = g(x) + g(y), \]
\[ |\alpha_n - \alpha| \to 0 \quad \text{and} \quad g(x_n - x) \to 0 \quad \text{implies} \quad g(\alpha_n x_n - \alpha x) \to 0. \]

Recall that a linear topological space \( X \) over the real field \( \mathbb{R} \) with a paranorm obeying these rules (i)-(iv) is called a paranormed space. From now on, let’s assume that \( (p_k) \) be a bounded sequence of strictly positive real numbers with \( \sup_k p_k = H \) and \( M = \text{max} \{1, H\} \). Then, the linear spaces \( \ell_\infty(p) \), \( c(p) \) and \( c_0(p) \) were defined by Maddox in [1] (see also Simons [2] and Nakano [3]) as follows:

\[ \ell_\infty(p) = \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\}, \]
\[ c(p) = \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} |x_k - l|^{p_k} = 0 \quad \text{for some} \quad l \in \mathbb{R} \right\}, \]
\[ c_0(p) = \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} |x_k|^{p_k} = 0 \right\}, \]

which are the complete spaces paranormed by

\[ g(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/M}. \]

Throughout the paper, it will be assumed that \( p_k^{-1} + (p_k')^{-1} = 1 \) provided that \( 1 < \inf p_k \leq H < \infty \).

The set of all sequences \( u \) such that \( u_k \neq 0 \) for all \( k \in \mathbb{N} \) will be denoted by \( U \). For \( u \in U \), let \( 1/u = (1/u_k) \).

For arbitrary sequence spaces \( X \) and \( Y \), the set \( S(X, Y) \) defined by

\[ S(X, Y) = \{ z = (z_k) \in \omega : xz = (x_kz_k) \in Y \quad \text{for all} \quad x \in X \}, \]
is called the multiplier space of \( X \) and \( Y \). With the notation of (1.1), the alpha-, beta- and gamma-duals of a sequence space \( X \) which are respectively denoted by \( X^\alpha \), \( X^\beta \) and \( X^\gamma \), are defined by

\[
X^\alpha = S(X, \ell_1), \quad X^\beta = S(X, c_0) \quad \text{and} \quad X^\gamma = S(X, bs).
\]

The \( f \)-dual \( \lambda^f \) of a sequence space of \( \lambda \) is defined as \( \lambda^f = \{ (f(c^n)) : f \in \lambda' \} \), where \( \lambda' \) denotes the continuous dual of the space \( \lambda \).

Any given sequence space \( X \) paranormed by \( h \) contains a sequence \( (a_n) \) with the property that for every \( x \in X \) there is a unique sequence of scalars \( (\alpha_n) \) such that

\[
\lim_{n \to \infty} h \left( x - \sum_{k=0}^{n} \alpha_k a_k \right) = 0
\]

then \( (a_n) \) is called a Schauder basis (or basis) for \( X \). The series \( \sum_k \alpha_k a_k \) which has the sum \( x \) is then called the expansion of \( x \) with respect to \( (a_n) \) and written as \( x = \sum_k \alpha_k a_k \).

In this paragraph, we shall introduce the notion of a matrix transformation from \( X \) to \( Y \). Let \( X, Y \) be any two sequence spaces. Given any infinite matrix \( A = (a_{nk}) \) of real numbers \( a_{nk} \), where \( n, k \in \mathbb{N} \), any sequence \( x \), we write \( Ax = (Ax)_n \), the \( A \)-transform of \( x \), if \( (Ax)_n = \sum_k a_{nk} x_k \) converges for each \( n \in \mathbb{N} \). For simplicity in notation, here and in what follows, the summation without limits runs from 0 to \( \infty \). If \( x \in X \) implies that \( Ax \in Y \) then we say that \( A \) defines a matrix mapping from \( X \) into \( Y \) and denote it by \( A : X \to Y \). By \( (X : Y) \), we mean the class of all infinite matrices such that \( A : X \to Y \).

The layout all of the present article is as follows:

At the beginning of the Section 2; some historical developments related to this matter are given before the paranormed sequence spaces \( Z(A_\lambda) \) of non-absolute type which are the set of all sequences whose \( A_\lambda \)-transforms are in the spaces \( Z(p) \) and then their alpha-, beta- and gamma-duals are computed. In addition to this, the basis of the spaces \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \) are obtained. In the final section of the paper, the dual summability methods of the new sort are defined and an analysis about these types methods is given.

2. The Paranormed Sequence spaces \( Z(p, A_\lambda) \) of Non-absolute Type

We start with this section by defining the sequence spaces \( \ell_\infty(p, A_\lambda) \), \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \) of non-absolute type obtained by the domain of \( A_\lambda \) in the Maddox’s spaces \( \ell_\infty(p) \), \( c(p) \) and \( c_0(p) \), and to prove that these are the complete paranormed linear spaces, and determine their alpha-, beta- and gamma-duals. Additionally, we give the basis for the spaces \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \).

The matrix domain \( X_A \) of an infinite matrix \( A \) for any sequence space \( X \) is described as

\[
X_A = \{ x = (x_k) \in \omega : Ax \in X \}.
\]

The new sequence space \( X_A \) generated by an infinite matrix \( A \) from the space \( X \) either includes the space \( X \) or is included by the space \( X \), in general, i.e., the space \( X_A \) is the expansion or the contraction of the original space \( X \). The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed. For instance, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. For more detail on the domains of some triangle matrices in certain sequence spaces, the reader may refer to Başar (see [15, p. 50]).

Here, we give some historical development about these subjects. Malkowsky and Savaş [16], and Choudhary and Mishra [17] have defined the sequence spaces \( Z(u, v; X) \) which consist of all sequences such that their \( G(u, v) \) and \( S \)-transforms are in \( X \in \{ \ell_\infty, c, c_0, \ell_p \} \) and \( \ell(p) \), respectively; where \( u, v \in \mathcal{U} \) and \( G(u, v) = (g_{nk}) \) and \( S = (s_{nk}) \) are defined by

\[
g_{nk} = \begin{cases} u_n v_k & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}, \quad \text{and} \quad s_{nk} = \begin{cases} 1 & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}
\]
for all \( k, n \in \mathbb{N} \). The set \( bs(p) \) of all series whose sequences of partial sums are in \( \ell_\infty(p) \), which is formerly defined in [18] and investigated by Başar and Altay [19]. Recently, Altay and Başar [20] have studied the space \( r^s(p) \) which consists of all sequences whose \( R^s \)-transforms are in the space \( \ell(p) \). With the notation of (2.1), the spaces \( Z(u, v; X) \), \( \ell(p) \), \( bs(p) \) and \( r^s(p) \) can be redefined by

\[
Z(u, v; X) = X_{C(u, v)} \cup \ell(p) = \{\ell(p)\}_{S}, \quad bs(p) = \{\ell_\infty(p)\}_{S} \quad \text{and} \quad r^s(p) = \{\ell(p)\}_{RS}.
\]

Altay and Başar [21] have recently examined the spaces \( X(u, v; p) = \{X(p)\}_{C(u, v)} \), where \( X \in \{\ell_\infty(c, c_0)\} \). From now on, let’s assume that \( \lambda = (\frac{\lambda_j}{\lambda_{j-1}})_{j=0}^\infty \) be a strictly increasing sequence of positive reals tending to infinity, that is

\[
0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty.
\]

Then, we define the infinite matrix \( A_\lambda = (a_{nj}(\lambda)) \) for all \( n, j \in \mathbb{N} \) by

\[
a_{nj}(\lambda) = \begin{cases} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_n - \lambda_{n-1}} , & (0 \leq j \leq n) \\ \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_n - \lambda_{n-1}} , & (j > n) \end{cases}
\]

It is clear that the matrix \( A_\lambda \) is triangle, that is \( a_{nn}(\lambda) \neq 0 \) and \( a_{nk}(\lambda) = 0 \) for all \( k > n \), where \( k, n \in \mathbb{N} \). Following Başar and Altay [19], Altay and Başar [20, 21] and Mursaleen and Noman [22, 23], we want to identify the sequence spaces \( Z(p, A_\lambda) \) by

\[
Z(p, A_\lambda) = \left\{ x = (x_k) \in \omega : y = \left( \sum_{j=0}^{n} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_n - \lambda_{n-1}} x_j \right) \in Z(p) \right\}.
\]

Here and after, we assume \( Z \in \{\ell_\infty, c, c_0\} \) and use the convention that any term with a negative subscript is equal to zero, namely, \( \lambda_{-2} = \lambda_{-1} = 0 \) and \( x_{-1} = 0 \).

If \( p_k = 1 \) for every \( k \in \mathbb{N} \), we write \( Z(A_\lambda) \) for \( Z(p, A_\lambda) \) which are introduced by Braha and Başar [24]. If \( X \) is any normed or paranormed sequence space then we call the matrix domain \( X_{A_\lambda} \) as the Generalized weighted means (GWM) sequence space. It is natural that these spaces can also be defined with the notation of (2.1) that

\[
Z(p, A_\lambda) = \{Z(p)\}_{A_\lambda}.
\]

Define the sequence \( y = (y_k) \), which will be frequently used, by the \( A_\lambda \)-transform of a sequence \( x = (x_k) \), i.e.,

\[
y_n = \sum_{j=0}^{n} \frac{\Delta^2 \lambda_j}{\lambda_n - \lambda_{n-1}} x_j , \quad (n \in \mathbb{N}),
\]

where \( \Delta^2 \lambda_j = \Delta(\Delta \lambda_j) = \Delta(\lambda_j - \lambda_{j-1}) = \lambda_j - 2\lambda_{j-1} + \lambda_{j-2} \). Although Theorems 2.1, 2.2 and 2.3 below, are related to the sequence spaces \( \ell_\infty(p, A_\lambda) \), \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \), we give the proof only for one of those spaces. We leave the details to the reader because the proof can be obtained similarly for other spaces. Now, we may begin with the following theorem which is essential in the study.

**Theorem 2.1.** The following statements hold:

(a) \( \ell_\infty(p, A_\lambda) \), \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \) are the complete linear metric spaces paranormed by \( h \), defined by

\[
h(x) = \sup_{n \in \mathbb{N}} \left( \sum_{j=0}^{n} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_n - \lambda_{n-1}} |x_j| \right)^{p_n/M}.
\]

\( h \) is a paranorm for the spaces \( \ell_\infty(p, A_\lambda) \) and \( c(p, A_\lambda) \) only in the trivial case \( \inf p_k = 0 \) when \( \ell_\infty(p, A_\lambda) = \ell_\infty(A_\lambda) \), \( c(p, A_\lambda) = c(A_\lambda) \).

(b) The sets \( Z(A_\lambda) \) are the Banach spaces with \( \|x\|_{Z(A_\lambda)} = \|y\|_{Z} \).

**Proof.** To verify this claim, a certain amount of functional analysis will be used. As they are similar to each other, we will only prove theorem for the space \( c_0(p, A_\lambda) \), using a standard type procedure. The following inequalities satisfied for \( s = (s_j) \), \( x = (x_j) \in c_0(p, A_\lambda) \) (see [25, p. 30]) and we easily obtain
linearity of \( c_0(p, A_\lambda) \) with respect to the coordinatewise addition and scalar multiplication

\[
\sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \frac{\Delta^2 \lambda_j}{\lambda_n - \lambda_{n-1}} (s_j + x_j) \right|^{p_n/M} \leq \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \frac{\Delta^2 \lambda_j}{\lambda_n - \lambda_{n-1}} s_j \right|^{p_n/M} + \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \frac{\Delta^2 \lambda_j}{\lambda_n - \lambda_{n-1}} x_j \right|^{p_n/M}
\]

for any \( \alpha \in \mathbb{R} \) (see [1])

\[
|\alpha|^{p_n} \leq \max \{1, |\alpha|^M\}.
\]

It is clear that \( h(0) = 0 \) and \( h(x) = h(-x) \) for all \( x \in c_0(p, A_\lambda) \). One more time, the inequalities (2.3) and (2.4) result in the subadditivity of \( h \) and

\[
h(\alpha x) \leq \max \{1, |\alpha|\} h(x).
\]

Let’s assume that \( (x^k) \) be any sequence of the points lying in \( c_0(p, A_\lambda) \) such that \( h(x^k - x) \to 0 \) and \( (\alpha_k) \) also be any sequence of scalars such that \( \alpha_k \to \alpha \) as \( k \to \infty \). Thus, since the inequality

\[
h(\alpha_k x^k - \alpha x) \leq h(x) + h(x^k - x)
\]

holds by subadditivity of \( h \), \( \{h(x^k)\} \) is bounded and therefore we obtain

\[
h(\alpha_k x^k - \alpha x) = \sup_{n \in \mathbb{N}} \left| \sum_{j=0}^{n} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_n - \lambda_{n-1}} (\alpha_k x_j^{(k)} - \alpha x_j) \right|^{p_n/M}
\]

which tends to zero as \( k \to \infty \). This means that the scalar multiplication is continuous. As a conclusion, \( h \) is a paranorm on the space \( c_0(p, A_\lambda) \).

Now, if we prove the completeness of the space \( c_0(p, A_\lambda) \) then the proof ends. Let’s suppose that \( (x^i) \) be any arbitrary Cauchy sequence in the space \( c_0(p, A_\lambda) \), where \( x^i = (x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots) \). In that case, there exists a positive integer \( n_0(\varepsilon) \) such that

\[
h(x^i - x^j) < \varepsilon
\]

for all \( i, j \geq n_0(\varepsilon) \) for any given \( \varepsilon > 0 \). By using definition of \( h \), for each fixed \( k \in \mathbb{N} \) we get

\[
(A_{\lambda} x^i)^k - (A_{\lambda} x^j)^k \leq \sup_{k \in \mathbb{N}} \left| (A_{\lambda} x^i)^k - (A_{\lambda} x^j)^k \right|^{p_k/M} < \varepsilon, \quad (i, j \geq n_0(\varepsilon))
\]

This newly obtained formula results in the fact that \( \{(A_{\lambda} x^0)_k, (A_{\lambda} x^1)_k, (A_{\lambda} x^2)_k, \ldots\} \) is a Cauchy sequence of real numbers for every fixed \( k \in \mathbb{N} \). Since \( \mathbb{R} \) is complete, it converges, say \( (A_{\lambda} x^i)^k \to (A_{\lambda} x)^k \) as \( i \to \infty \). Using these infinitely many limits \( (A_{\lambda} x)^0, (A_{\lambda} x)^1, (A_{\lambda} x)^2, \ldots \), we define the sequence \( \{(A_{\lambda} x)^0, (A_{\lambda} x)^1, (A_{\lambda} x)^2, \ldots\} \). We have from (2.5) for every fixed \( k \in \mathbb{N} \) with \( j \to \infty \) we easily obtain

\[
(A_{\lambda} x^i)^k - (A_{\lambda} x)^k \leq \varepsilon, \quad (i \geq n_0(\varepsilon)).
\]

Since \( x^i = (x_0^{(i)}) \in c_0(p, A_\lambda) \),

\[
(A_{\lambda} x^i)^k \leq \varepsilon
\]

for all \( k \in \mathbb{N} \). Thus, we have by (2.6) that

\[
(A_{\lambda} x)^k \leq \varepsilon
\]

This indicates that the sequence \( A_{\lambda} x \) belongs to the space \( c_0(p, A_\lambda) \). As \( (x^i) \) was an arbitrary Cauchy sequence, it follows that the space \( c_0(p, A_\lambda) \) is complete. This conclusion is what was sought for.

It can easily be controlled that the absolute property is invalid on the spaces \( \ell_{\infty}(p, A_\lambda) \), \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \) in other words, \( h_i(x) \neq h_i(|x|) \) for at least one sequence found in the spaces \( \ell_{\infty}(p, A_\lambda) \), \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \). Thus, we can say that \( \ell_{\infty}(p, A_\lambda) \), \( c(p, A_\lambda) \) and \( c_0(p, A_\lambda) \) are the sequence spaces of non-absolute type, in which \( |x| = (|x_k|) \).
Now, let us give the definition of the isomorphism. A bijective linear transformation \( \tau : X \to Y \) is called an isomorphism from \( X \) to \( Y \). When an isomorphism from \( Y \) to \( X \) exist, we say that \( X \) to \( Y \) are isomorphic and write \( X \cong Y \).

**Theorem 2.2.** When \( 0 < p_k \leq H < \infty \), the GWM sequence spaces \( \ell_\infty^p(p, A) \), \( c(p, A) \) and \( c_0(p, A) \) of non-absolute type are linearly isomorphic to those spaces \( \ell_\infty(p) \), \( c(p) \) and \( c_0(p) \), respectively.

**Proof.** Because the second and third parts of the theorem have exactly same idea as in the first part of it, the proof of those parts can be obtained in a similar way. To prove the fact that \( \ell_\infty^p(p, A) \cong \ell_\infty \), we must guarantee the existence of a linear bijection between the spaces \( \ell_\infty^p(p, A) \) and \( \ell_\infty(p) \) for \( 1 \leq p_k \leq H < \infty \), due to the definition of linear isomorphism. Let us take into consideration the transformation \( \tau \) defined above, using the notation of (2.2) from \( \ell_\infty^p(p, A) \) to \( \ell_\infty(p) \) by \( x \mapsto y = \tau x \). The linearity of the transformation \( \tau \) is obvious. Moreover, it is clear that \( x = \theta \) whenever \( \tau x = \theta \), which results in the fact that \( \tau \) is injective.

For any arbitrary sequence \( y \) lying in \( \ell_\infty(p) \), let’s define the sequence \( x = (x_n) \) by

\[
   x_n = \sum_{j=n-1}^{n-1} (-1)^{n-j} \frac{\lambda_j - \lambda_{j+1}}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} y_j ; \quad (n \in \mathbb{N}).
\]

Now, we can easily write the following steps, using the newly obtained sequence \( x = (x_n) \)

\[
   h(x) = \sup_{n \in \mathbb{N}} \left| \frac{\lambda_j - 2\lambda_{j+1} + \lambda_{j+2}}{\lambda_n - \lambda_{n-1}} x_j \right|^{p_n/M} = \sup_{n \in \mathbb{N}} |y_n|^{p_n/M} = g(y) < \infty.
\]

From the above equations, we understand that there exists an \( x \in \ell_\infty^p(p, A) \) for every \( y \in \ell_\infty(p) \), in a more clear way, \( \tau \) is surjective and its paranorm is maintained. Since \( \tau \) is both injective and subjective, \( \tau \) is a linear bijection. As a result, this shows that \( \ell_\infty^p(p, A) \) and \( \ell_\infty(p) \) are linearly isomorphic. In fact, this concludes our proof.

For the sake of simplicity, here and in what follows, it will be assumed that the summation without limits runs from 0 to \( \infty \).

**Theorem 2.3.** Let \( \{1/(\lambda_n - \lambda_{n-1})\} \in \mathcal{C}_0 \). The sequence space \( c_0(A) \) has AD property.

**Proof.** It is a fact that if \( f \in [c_0(A)]' \), then in that case \( f(x) = g(Bx) \) for some \( g \in \mathcal{C}_0 = \ell_1 \). Because of the fact that \( \mathcal{C}_0 \) has \( \Lambda K \) property and \( \mathcal{C}_0 \cong \ell_1 \),

\[
   f(x) = \sum_i b_i(A \lambda^i x) = (A \lambda^i x)_i
\]

for some values of \( b = (b_i) \in \ell_1 \). Again due to the fact that \( \{1/(\lambda_n - \lambda_{n-1})\} \in \mathcal{C}_0 \) according to the hypothesis, the inclusion \( \phi \subset c_0(A) \) valid. For any given \( f \in [c_0(A)]' \) and \( e^k \in \phi \), we have

\[
   f(e^k) = \sum_i b_i(A \lambda^i e^k) = \{A(\lambda^i b) \}_{k} ; \quad (k \in \mathbb{N})
\]

here \( A' \) denotes the transpose of the matrix \( A \lambda \). Since Hahn-Banach Theorem requires that \( \phi \subset c_0(A) \) is dense in \( c_0(A) \) iff \( \phi = \phi \) for \( b \in \ell_1 \); this implies the condition \( b = \theta \).

Because of the fact that the null space of the operator \( A(\lambda) \) on \( \omega \) is \( \{\theta\} \), \( c_0(A) \) is said to have AD property.

**Lemma 2.4.** [26, pp. 106, 108] Let \( \lambda \) be an FK-space which contains \( \phi \). Then, the following statements hold:

(i) \( \lambda^\omega \subset \lambda^\beta \subset \lambda^\gamma \).
(ii) If \( \lambda \) has AD then \( \lambda^\beta = \lambda^\gamma \).
(iii) \( \lambda \) has AD iff \( \lambda^\beta = \lambda^\gamma \).

Now, as an easy consequence of Theorem 2.3 and Part (iii) of Lemma 2.4, we have
Corollary 2.5.

\[
\left\{ \{c_0\}, A_{\lambda} \right\}^c = \left\{ \{c_0\}, A_{\lambda} \right\}^c = \left\{ b = (b_k) \in \omega : \sum_k \left| \Delta \lambda_k \sum_{j=k}^{k+1} (-1)^{k-j} \frac{b_j}{\Delta^2 \lambda_j} \right| < \infty \right\}.
\]

Theorem 2.6. Let \(u, v \in U\) and \(z = (z_k) \in \omega\) be any sequence. Define the sequences \(\delta = (\delta_k), \sigma = (\sigma_k)\) and the matrix \(B = (b_{nk})\) by

\[
\delta_k = (\lambda_k - \lambda_{k-1}) \left( \frac{z_{k+1}}{\lambda_k - 2 \lambda_{k-1} + \lambda_{k-2}} - \frac{z_k}{\lambda_k + 2 \lambda_{k-1} + \lambda_{k-2}} \right), \quad \sigma_k = z_k \left( \frac{\lambda_k - \lambda_{k-1}}{\lambda_k - 2 \lambda_{k-1} + \lambda_{k-2}} \right)
\]

and

\[
(2.7) \quad b_{nk} = \begin{cases} 
\delta_k, & (k < n) \\
\sigma_k, & (k = n) \\
0, & (k > n)
\end{cases}, \quad (n, k \in \mathbb{N}).
\]

Then,

\[
(Z(p, A_{\lambda}))^c = \{ z = (z_k) \in \omega : B \in (Z(p) : \ell_\infty) \}.
\]

and

\[
(Z(p, A_{\lambda}))^c = \{ z = (z_k) \in \omega : B \in (Z(p) : c) \}
\]

Proof. Consider the equality

\[
(2.8) \quad \sum_{k=0}^{n} z_k x_k = \sum_{k=0}^{n-1} \delta_k y_k + \sigma_n y_n = (B y)_n, \quad (n \in \mathbb{N});
\]

where \(B = (b_{nk})\) defined by (2.7).

Thus, one can easily deduce from (2.8) that \(z x = (z_k x_k) \in cs\) or \(bs\) whenever \(x = (x_k) \in Z(p, A_{\lambda})\) if and only if \(B y \in c\) or \(\ell_\infty\) whenever \(y = (y_k) \in Z(p)\). This means that \(z = (z_k) \in [Z(p, A_{\lambda})]^c\) or \(z = (z_k) \in [Z(p, A_{\lambda})]^c\) if and only if \(B \in (Z(p) : c)\) or \(B \in (Z(p) : \ell_\infty)\) which is what we wished to prove. This completes the proof.

It is clear that the technique is very useful in the calculation of beta- and gamma-duals of the sequence spaces derived by an infinite matrix from a sequence space. Therefore, as an application of Theorem 2.6, we have:

Corollary 2.7. Let \(u, v \in U\) and \(\mu = (\mu_k)\) with \(\mu_k = M^{1/k}\) for all \(k \in \mathbb{N}\). Then,

(i) \([\ell_\infty(p, A_{\lambda})]^c = \bigcap_{M \geq 1} \{ t = (t_k) \in \omega : \delta \mu \in \ell_1, \sigma \mu \in \ell_\infty, \tau \mu \in c_0 \} \}.

(ii) \([c(p, A_{\lambda})]^c = \bigcup_{M \geq 1} \{ t = (t_k) \in \omega : \delta \mu \in \ell_1, \sigma \mu \in \ell_\infty, \tau \mu \in c_0 \} \}

(iii) \([c_0(p, A_{\lambda})]^c = \bigcup_{M \geq 1} \{ t = (t_k) \in \omega : \delta \mu \in \ell_1, \sigma \mu \in \ell_\infty, \tau \mu \in c_0 \} \}

(iv) \([\ell_\infty(p, A_{\lambda})]^c = \bigcap_{M \geq 1} \{ t = (t_k) \in \omega : \delta \mu \in \ell_1, \sigma \mu \in \ell_\infty, \tau \mu \in c_0 \} \}

(v) \([c(p, A_{\lambda})]^c = \bigcup_{M \geq 1} \{ t = (t_k) \in \omega : \delta \mu \in \ell_1, \sigma \mu \in \ell_\infty, \tau \mu \in c_0 \} \}

(vi) \([c_0(p, A_{\lambda})]^c = \bigcup_{M \geq 1} \{ t = (t_k) \in \omega : \delta \mu \in \ell_1, \sigma \mu \in \ell_\infty, \tau \mu \in c_0 \} \}

Now, we may give a sequence of the points of the spaces \(c_0(p, A_{\lambda})\) and \(c(p, A_{\lambda})\) which forms a Schauder basis for those spaces. Because of the isomorphism \(T\) between the sequence spaces \(c_0(p, A_{\lambda})\) and \(c(p, A_{\lambda})\) and \(c(p)\) is onto, the inverse image of the basis of spaces \(c_0(p)\) and \(c(p)\) is the basis of the our new spaces \(c_0(p, A_{\lambda})\) and \(c_0(p)\), respectively. Therefore, we have:
Theorem 2.8. Let \( \lambda_k = (A_\lambda x)_k \) and \( 0 < p_k \leq H < \infty \) for all \( k \in \mathbb{N} \). Define the sequence \( b^{(k)} = \{ b_n^{(k)} \} \) for every fixed \( k \in \mathbb{N} \) by

\[
 b_n^{(k)} = \begin{cases} 
 (-1)^{n-k} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{n+1} - 2\lambda_{n-1} + \lambda_{n-2}}, & (k \leq n \leq k+1) \\
 0, & (0 \leq n < k \text{ or } n > k + 1)
\end{cases}
\]

Then, we have

(a) The sequence \( \{ b^{(k)} \} \) is a basis for the spaces \( c_0(p, A_\lambda) \), any \( x \) in \( c_0(p, A_\lambda) \) has a unique representation of the form \( x = \sum_k \lambda_k b^{(k)} \).
(b) The set \( \{ t, b^{(k)} \} \) is a basis for the space \( c(p, A_\lambda) \) and any \( x \in c(p, A_\lambda) \) has a unique representation of the form \( x = t + \sum_k [\lambda_k - t_k] b^{(k)} \), where \( t = (t_k) \) with \( t_k = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j - \lambda_{j-1}}{\lambda_k - 2\lambda_{j-1} + \lambda_{j-2}} \) for all \( k \in \mathbb{N} \) and \( t = \lim_{k \to \infty} (A_\lambda x)_k \).

3. Some Matrix Mappings Related to the Sequence Spaces \( Z(p, A_\lambda) \)

Here, we define a pair of summability methods. The first one will be applied to those sequences in the space \( Z(p, A_\lambda) \). The second one will be applied to those sequences in the space \( Z(p) \). Then, as a result, we will purpose a basic theorem related to this type summability methods. Moreover, a class \( (Z(p, A_\lambda), \mu) \) of infinite matrices will be characterized for \( \mu \in (\ell_\infty, c) \). Later, several other classes from those ones will be derived using an appropriate relation. For the sake of brevity, from now on, we will use the following shorthands

\[
 a_{nk} = \Delta \lambda_k \left( \frac{a_{nk}}{\Delta^2 \lambda_k} + \frac{a_{nk+1}}{\Delta^2 \lambda_{k+1}} \right) \quad \text{and} \quad b_{nk} = \sum_{j=0}^{n} \Delta^2 \lambda_j \frac{a_{nk+1}}{\Delta \lambda_k} \frac{a_{nk+1}}{\Delta \lambda_k} 
\]

for all \( k, n \in \mathbb{N} \) and the \( n^{th} \) row of the infinite matrix \( A = (a_{nk}) \) is going to be defined by \( A_n \). The other notations will be similar to those widely used in other manuscripts. Lastly, it is supposed that a term having negative subscript will be equal to naught.

We assume that the infinite matrices \( E = (e_{nk}) \) and \( F = (f_{nk}) \) map the sequences \( x = (x_k) \) and \( y = (y_k) \), respectively. These sequences are connected by Eq. (2.2) to the sequences \( s = (s_n) \) and \( t = (t_n) \), respectively. That is,

\[
s_n = (Ex)_n = \sum_k e_{nk} x_k , \quad (n \in \mathbb{N}),
\]

\[
t_n = (Fy)_n = \sum_k f_{nk} y_k , \quad (n \in \mathbb{N}).
\]

It is obvious that while the method \( F \) is applied to the \( A_\lambda \)-transform of the sequence \( x = (x_k) \), the method \( E \) is directly applied to those terms of the sequence \( x = (x_k) \). In fact, the methods \( E \) and \( F \) are basically different.

The fact that the matrix product \( FA_\lambda \) exists is a weaker assumption than the one indicating the matrix \( F \) belonging to any matrix class generally. It is assumed that the matrix product \( FA_\lambda \) exists, a much weaker assumption than the conditions on the matrix \( F \) which belongs to any matrix class generally. In this situation, the methods \( E \) and \( F \) in (3.1) and (3.2) are said to be the pair of summability methods, in short PSM, if \( s_n \) becomes \( t_n \) or vice versa with the assumption that the application of the formal summation by parts. Thus, we conclude that \( FA_\lambda \) exists and equals to \( E \) and \( (FA_\lambda)x = FA_\lambda x \) is valid, if one of the sides exists. This sentence is equal to the following relation

\[
(3.3) e_{nk} = \sum_{j=k}^{\infty} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_{j-3} - \lambda_{j-1}} f_{nj} \quad \text{or} \quad f_{nk} = (\lambda_k - \lambda_{k-1}) \left( \frac{e_{nk}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} - \frac{e_{n,k+1}}{\lambda_{k+1} - 2\lambda_k + \lambda_{k-1}} \right) 
\]

for all \( k, n \in \mathbb{N} \).

It is time to give a brief analysis on the PSM. It is seen that \( t_n \) reduces to \( s_n \) by the following chain of equation,
\[ t_n = \sum_k f_{nk} y_k = \sum_k f_{nk} \left( \sum_{j=0}^{k} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} x_j \right) = \sum_j \left( \sum_{k=j}^{\infty} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_k - \lambda_{k-1}} f_{nk} \right) x_j = s_n. \]

Since the order of summation cannot be reversed, it is not necessary that the methods \( F \) and \( E \) are equivalent.

In (3.1) and (3.2) the right hand side partial sums of series are connected with the following relation

\[ (3.4) \sum_{k=0}^{m} e_{nk} x_k = \sum_{k=0}^{m-1} \Delta \lambda_k \left( \frac{e_{nk} - e_{n,k+1}}{\Delta^{2} \lambda_k} \right) y_k + \frac{\lambda_m - \lambda_{m-1}}{\lambda_m - 2\lambda_{m-1} + \lambda_{m-2}} e_{nm} y_m; \quad (m, n \in \mathbb{N}). \]

If, any one of the series on the right hand side of (3.1) and (3.2) converges for a given \( n \in \mathbb{N} \), then the other one also converges iff

\[ \lim_{m \to \infty} \frac{\lambda_m - \lambda_{m-1}}{\lambda_m - 2\lambda_{m-1} + \lambda_{m-2}} e_{nm} y_m = z_n \]

for each fixed \( n \in \mathbb{N} \). Let's assume \( m \to \infty \), then from (3.4) we have

\[ s_n = t_n + z_n; \quad (m \in \mathbb{N}). \]

Therefore, if \( (y_n) \) is summable according to either method \( F \) or \( E \), then it is also summable by the other one iff (3.5) is valid and

\[ \lim_{n \to \infty} z_n = \alpha. \]

It is seen that the limits of \( (s_n) \) and \( (t_n) \) are different by \( \alpha \). Thus, any sequence summable's the \( F \)- and \( E \)-limits by one of those agree iff when \( \alpha = 0 \) \( F \) summability implies (3.6) is valid, vice versa when \( F \) and \( E \) changed. It concludes that \( E \) and \( F \) are inconsistent methods when \( \alpha \neq 0 \) and (3.6) is valid.

After the above analysis, now we can give the following fundamental theorem about the matrix mappings on the sequence space \( Z(p, A_\lambda) \).

**Theorem 3.1.** Let's assume that \( E = (e_{nk}) \) and \( F = (f_{nk}) \) are connected to each other in view of relation (3.3) and \( \mu \) is any given sequence space. In that case, for all \( k, m, n \in \mathbb{N} \), \( E \in (Z(p, A_\lambda) : \mu) \) iff \( F \in (Z(p) : \mu) \) and

\[ (3.7) \quad f_{mk}^{(n)} \in (Z(p) : e) \]

\[ f_{mk}^{(n)} = \begin{cases} (\lambda_k - \lambda_{k-1}) \left( \frac{e_{nk}}{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}} - \frac{e_{n,k+1}}{\lambda_{k+1} - 2\lambda_k + \lambda_{k-1}} \right), & (k < m) \\ \frac{\lambda_m - \lambda_{m-1}}{\lambda_m - 2\lambda_{m-1} + \lambda_{m-2}} e_{nm}, & (k = m) \\ 0, & (k > m) \end{cases} \]

**Proof.** Let's assume that \( E \) and \( F \) be PSM, in other words, (3.3) is valid, \( \mu \) is a sequence space and \( Z(p, A_\lambda) \) and \( Z(p) \) are linearly isomorphic.

Again, let's assume that \( E \in (Z(p, A_\lambda) : \mu) \) and \( y \in Z(p) \). In that case, there exists \( FA_\lambda \) and \( (e_{nk})_{k \in \mathbb{N}} \in [Z(p, A_\lambda)]^\beta \) resulting in \( (f_{nk})_{k \in \mathbb{N}} \in Z^\beta(p) \) for every \( n \in \mathbb{N} \). Thus, for each \( y \in Z(p) \) there exists a \( Fy \), and if we take \( m \to \infty \) in the following equality

\[ \sum_{k=0}^{m} f_{nk} y_k = \sum_{k=0}^{m} \sum_{j=k}^{\infty} \frac{\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}}{\lambda_j - \lambda_{j-1}} f_{mj} x_j; \quad (m, n \in \mathbb{N}), \]

we have the fact that \( Fy = Ex \) using (3.3) and this leads us to the conclusion \( F \in (Z(p) : \mu) \).

In the other way around, let's assume that \( F \in (Z(p) : \mu) \) and (3.7) is valid, and \( x \in Z(p, A_\lambda) \). In that case, for every \( n \in \mathbb{N} \), we have that \( (f_{nk})_{k \in \mathbb{N}} \in Z^\beta(p) \) resulting in \( (e_{nk})_{k \in \mathbb{N}} \in [Z(p, A_\lambda)]^\beta \) together with (3.7). Thus we can say that \( Ex \) exists. If we take \( m \to \infty \) and use (3.4), we obtain \( Ex = Fy \) and this clearly indicates the fact that \( E \in (Z(p, A_\lambda) : \mu) \). This is what we want to prove.

If we change the places of the spaces \( Z(p, A_\lambda) \) and \( Z(p) \) with \( \mu \), then we obtain
Theorem 3.2. Let’s assume that for all $k, n \in \mathbb{N}$ and for any given sequence space $\mu$, the elements of the infinite matrices $R = (r_{nk})$ and $Q = (q_{nk})$ are connected to each other with the following relationship

$$q_{nk} = \sum_{j=k}^{m} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_j - \lambda_{j-1}} r_{jk}$$

(3.8)

In that case, $R \in (\mu : Z(p, A_\lambda))$ iff $Q \in (\mu : Z(p))$.

Proof. Now, let’s suppose that $z = (z_k) \in \mu$ and take into account the following equality together with (3.8)

$$\sum_{k=0}^{m} q_{nk}z_k = \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{\lambda_j - 2\lambda_{j-1} + \lambda_{j-2}}{\lambda_j - \lambda_{j-1}} r_{jk}z_k , \quad (m, n \in \mathbb{N})$$

resulting in the fact that $(Qz)_n = [A_\lambda(Rz)]_n$ when $m \to \infty$ that $(Qz)_n = [A_\lambda(Rz)]_n$. Thus, it is obvious that $Rz \in Z(p, A_\lambda)$ when $z \in \mu$ iff $Qz \in Z(p)$ when $z \in \mu$. In fact, this is what we want to prove. \qed

Both Theorem 3.1 and Theorem 3.2 have several consequences with respect to the choice of the sequence space $\mu$. If we replace $E$ and $R$ by $F = EA_\lambda^{-1}$ and $Q = A_\lambda R$, we obtain the necessary and sufficient conditions for $(Z(p, A_\lambda) : \mu)$ and $(\mu : Z(p, A_\lambda))$ using Therefore by Theorem 3.1 and Theorem 3.2, respectively. The necessary and sufficient conditions on $F$ and $Q$ are taken from the related consequences in the literature.

The Grosse-Erdmann [27]'s necessary and sufficient conditions characterizing the matrix mappings among the sequence spaces $\ell_\infty(p)$, $c_0(p)$ and $c_0(p)$ has been used. It is time to quote our own theorems on the characterization of some matrix classes related to the sequence space $Z(p, A_\lambda)$. We assume that $N$ and $K$ denote the finite subsets of $\mathbb{N}$, $L$ and $M$ denote the natural numbers. Then the sets $K_1$ and $K_2$ are described as $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$, $K_2 = \{k \in \mathbb{N} : p_k > 1\}$. Finally, it is assume that $(q_n)$ is a non-decreasing bounded sequence of positive real numbers and the following conditions are taken into consideration:

$$\exists \ M, \sup_{K} \sum_{n} \left| \sum_{k \in K} a_{nk} M^{-1/p_k} \right|^{q_n} < \infty (q_n \geq 1)$$

(3.9)

$$\sum_{n} \left| \sum_{k} a_{nk} \right|^{q_n} < \infty (q_n \geq 1)$$

(3.10)

$$\forall \ M, \sup_{K} \sum_{n} \left| \sum_{k \in K} a_{nk} M^{1/p_k} \right|^{q_n} < \infty (q_n \geq 1)$$

(3.11)

$$\lim_{n \to \infty} |a_{nk}|^{q_n} = 0 \quad \text{for all } k$$

(3.12)

$$\forall \ L, \sup_{n \in N} \sup_{k \in K_1} |a_{nk} L^{1/q_n} |^{p_k} < \infty$$

(3.13)

$$\forall \ L, \exists \ M, \sup_{n \in N} \sum_{k \in K_2} |a_{nk} L^{1/p_n} M^{-1/p_k} |^{p_k} < \infty$$

(3.14)

$$\forall \ L, \exists \ M, \sup_{n \in N} L^{1/q_n} \sum_{k} |a_{nk}|^{M^{-1/p_k}} < \infty$$

(3.15)

$$\lim_{n \to \infty} \sum_{k} |a_{nk}|^{q_n} = 0$$

(3.16)
\(\forall M, \lim_{n \to \infty} \left( \sum_{k} |a_{nk}|M^{1/p_k} \right) = 0\)

\(\sup_{n \in \mathbb{N}} \sup_{k \in K_1} |a_{nk}|p_k < \infty\)

\(\exists M, \sup_{n \in \mathbb{N}} \sum_{k \in K_2} |a_{nk}M^{-1}|p_k < \infty\)

\(\exists (\alpha_k), \lim_{n \to \infty} |a_{nk} - \alpha_k|^{q_n} = 0 \text{ for all } k\)

\(\exists (\alpha_k), \forall L, \sup_{n \in \mathbb{N}} \sup_{k \in K_1} \left( |a_{nk} - \alpha_k|L^{1/q_n} \right)^{p_k} < \infty, \text{ for all } k \text{ and } q_n > 0\)

\(\exists (\alpha_k), \forall L, \exists M, \sup_{n \in \mathbb{N}} \sum_{k \in K_2} \left( |a_{nk} - \alpha_k|L^{1/q_n}M^{-1} \right)^{p_k} < \infty \text{ for all } k \text{ and } q_n > 0\)

\(\exists (\alpha_k), \forall L, \exists M, \sup_{n \in \mathbb{N}} \sum_{k} |a_{nk} - \alpha_k|L^{1/q_n}M^{-1/p_k} < \infty \text{ for all } k \text{ and } q_n > 0\)

\(\exists M, \sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}|M^{-1/p_k} < \infty\)

\(\exists \alpha, \lim_{n \to \infty} \left| \sum_{k} a_{nk} - \alpha \right|^{q_n} = 0, \quad (q_n > 0)\)

\(\exists (\alpha_k), \forall M, \lim_{n \to \infty} \left( \sum_{k} |a_{nk} - \alpha_k|M^{1/p_k} \right)^{q_n} = 0 \text{ for all } k \text{ and } q_n > 0\)

\(\forall M, \sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}|M^{1/p_k} < \infty\)

\(\exists L, \sup_{n \in \mathbb{N}} \sup_{k \in K_1} |a_{nk}L^{-1/q_n}|p_k < \infty, \quad (q_n > 0)\)

\(\exists L, \sup_{n \in \mathbb{N}} \sum_{k \in K_2} |a_{nk}L^{-1/q_n}|p_k < \infty, \quad (q_n > 0)\)

\(\exists M, \sup_{n \in \mathbb{N}} \left( \sum_{k} |a_{nk}|M^{-1/p_k} \right)^{q_n} < \infty, \quad (q_n > 0)\)

\(\sup_{n \in \mathbb{N}} \left( \sum_{k} a_{nk} \right)^{q_n} < \infty, \quad (q_n > 0)\)

\(\forall M, \sup_{n \in \mathbb{N}} \left( \sum_{k} |a_{nk}|M^{1/p_k} \right)^{q_n} < \infty, \quad (q_n > 0)\)
Corollary 3.3. (i) \( A \in \left( c_0(p, A_\lambda) : \ell(q) \right) \) if and only if (3.9) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = c_0 \).

(ii) \( A \in \left( c_0(p, A_\lambda) : c(q) \right) \) if and only if (3.20), (3.23) and (3.24) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = c_0 \).

(iii) \( A \in \left( c_0(p, A_\lambda) : \ell_\infty(q) \right) \) if and only if (3.30) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = c_0 \).

Corollary 3.4. (i) \( A \in \left( c(p, A_\lambda) : \ell(q) \right) \) if and only if (3.9) and (3.10) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = c \).

(ii) \( A \in \left( c(p, A_\lambda) : c(q) \right) \) if and only if (3.20), (3.23)–(3.25) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = c \).

(iii) \( A \in \left( c(p, A_\lambda) : \ell_\infty(q) \right) \) if and only if (3.30) and (3.31) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = c \).

Corollary 3.5. (i) \( A \in \left( \ell_\infty(p, A_\lambda) : \ell(q) \right) \) if and only if (3.11) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = \ell_\infty \).

(ii) \( A \in \left( \ell_\infty(p, A_\lambda) : c_0(q) \right) \) if and only if (3.17) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = \ell_\infty \).

(iii) \( A \in \left( \ell_\infty(p, A_\lambda) : c(q) \right) \) if and only if (3.26) and (3.27) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = \ell_\infty \).

(iv) \( A \in \left( \ell_\infty(p, A_\lambda) : \ell_\infty(q) \right) \) if and only if (3.32) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) and (3.7) also holds with \( Z = \ell_\infty \).

Corollary 3.6. (i) \( A \in \left( c_0(p) : c(q, A_\lambda) \right) \) if and only if (3.20), (3.23) and (3.24) hold with \( b_{nk} \) instead of \( a_{nk} \).

(ii) \( A \in \left( c_0(p) : \ell_\infty(q, A_\lambda) \right) \) if and only if (3.30) holds with \( b_{nk} \) instead of \( a_{nk} \).

Corollary 3.7. (i) \( A \in \left( c(p) : c(q, A_\lambda) \right) \) if and only if (3.20), (3.23)–(3.25) hold with \( b_{nk} \) instead of \( a_{nk} \).

(ii) \( A \in \left( c(p) : \ell_\infty(q, A_\lambda) \right) \) if and only if (3.30) and (3.31) hold with \( b_{nk} \) instead of \( a_{nk} \).

Corollary 3.8. (i) \( A \in \left( \ell_\infty(p) : c_0(q, A_\lambda) \right) \) if and only if (3.17) holds with \( b_{nk} \) instead of \( a_{nk} \).

(ii) \( A \in \left( \ell_\infty(p) : c(q, A_\lambda) \right) \) if and only if (3.26) and (3.27) hold with \( b_{nk} \) instead of \( a_{nk} \).

(iii) \( A \in \left( \ell_\infty(p) : \ell_\infty(q, A_\lambda) \right) \) if and only if (3.32) holds with \( b_{nk} \) instead of \( a_{nk} \).

Corollary 3.9. (i) \( A \in \left( \ell(p) : c_0(q, A_\lambda) \right) \) if and only if (3.12), (3.13) and (3.14) hold with \( b_{nk} \) instead of \( a_{nk} \).

(ii) \( A \in \left( \ell(p) : c(q, A_\lambda) \right) \) if and only if (3.18), (3.19), (3.20), (3.21) and (3.22) hold with \( b_{nk} \) instead of \( a_{nk} \).

(iii) \( A \in \left( \ell(p) : \ell_\infty(q, A_\lambda) \right) \) if and only if (3.28) and (3.29) hold with \( b_{nk} \) instead of \( a_{nk} \).

References


