A New Experimental and Numerical Investigation Using the Theta Time Scheme Combined with a Finite Element Spatial Approximation of the Soil-heat

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Abstract: In this paper, one-dimensional numerical model for simulation of the soil heat under conditions of evaporation was developed and validated by field experimental, and the governing equations were solved by the finite element spatial approximation approach combined with a theta time scheme, which predicts the temperature fields for different depths of a homogeneous ground and this for different times and days of a given month. We primarily interested parties in the methodology as well as its application to meteorology in Oran city in Algeria. However, this paper coated with oil could significantly improve the soil microclimatic for seed germination

Keywords: Finite element, Heart equation, Methodology data

1. Introduction

Mulching is very effective in modifying the soil microclimatic conditions for plant growth. In the last 20 years, the mulching effects of various kinds of plastic and natural materials have been tested in field experiments, especially for the effect on soil temperature (see [1, 2]). Computer modelling can provide a means of predicting the effects of mulching and, in many circumstances, can cut down drastically the amount of field test required. A one-dimensional numerical model for the prediction of
soil temperature beneath clear polyethylene (PE) film was presented by Mahrer (1979), and a two-
dimensional numerical model for the same prediction was presented later by Mahrer and Katan (1981).
A one-dimensional model for both soil water and heat behavior was presented by Mahrer (1984). The
results in his study showed that the moisture content of the mulched soil strongly affected the soil
temperature, and soil temperature was much lower at the edges of the mulch than at the middle. A one-
dimensional model for soil temperature regime under an asphalt emulsion film was developed by Chen
(1980), with the assumption that the film is in close contact with the soil surface. A one-dimensional
model suitable for describing both heat and moisture transfer through a surface residue-soil system was
also developed by Bristow and Campbell (1986). However, in that model the effect of the density and
the arrangement of the porous mulch was not considered. The numerical prediction is a young
discipline, since it is mainly developed during the second half of the twentieth century, benefiting from
ongoing progress by automatic calculation tools.

The techniques used can solve with numerical methods, the equations describing the behavior
of the atmosphere, that is to say to determine future values of its parameters characteristics starting
from known initial values through meteorological observations. The use of numerical computation is
required to solve these systems of nonlinear equations whose solutions cannot be determined
analytically in the general case. The construction of a numerical model of the atmosphere consists of
two distinct stages: the first is to establish a system of equations governing the behavior of continuous
atmosphere, called the second scan is to replace the equations on continuous variables by equations
involving discrete variable whose solutions are obtained by means of a suitable algorithm. The results
of a numerical weather prediction, solutions of discretized equations of dynamic meteorology,
therefore depend on the adopted scanning. In the late '40s methods of grid points were used to model
the universally large-scale atmospheric flow. During recent years, alternative methods have been used,
one of these methods is the finite element method. We present here the main tools for implementations
of the finite element method which belongs to the more general family of Galerkin methods. These are
commonly used instead of the finite difference method to treat the horizontal and vertical fields in
weather prediction models. Galerkin methods, which can solve numerically systems of partial
differential equations, do not directly use the field values at the points of a grid, but use series
expansions of functions suitably chosen so to be reduced to the solution of a system of ordinary
differential equations. There are two types of methods within this process: the finite element method
for which the functions are zero, except for a small area or they are equal to the low-order polynomials,
and the spectral method in which the functions are the functions of a spatial operator defined on the
whole area of the works.
The finite element method is one of the tools of applied mathematics. It is put in place, using principles inherited from the variational formulation or weak formulation, a discrete mathematical algorithm for finding an approximate solution of a partial differential equation (PDE or) on a compact domain with boundary conditions and / or in the interior of the compact. It speaks Dirichlet conditions (values at the edges) or Neumann (gradients at the edges). The finite element method is different than the spectral because it is not comprehensive, but rather determined by local values. However, it is distinct approximations gridded because the function is defined over the entire region and not just the discrete points.

As in the case of other numerical methods, questions arise as to the discretization:

* Existence and uniqueness of solutions;
* Stability;
* Convergence;
* And of course, the error measure between a discrete and a unique solution of the initial problem.

Our goal in this paper is to highlight the application of the method to an equation of elliptic type. In the practical part, we applied this method to the diffusion equation for the temperature in the ground for in Oran city in the west of Algeria.

2. Equation of the Diffusion Temperature in the Soil

In this section, we present a one-dimensional numerical prediction model that predicts the temperature field for different depths of soil homogeneous for different times and different days of a given month. By a finite element approximation of the Lagrange polynomial of order P1, determining the change in temperature in the soil over time by the knowledge of the parameters and characteristics of the soil. During a period of 24 hours, the profile of the surface temperature is substantially the response to radiation received diurnal. The inertia of the soil intervenes in depth. In this case, the soil parameters, among others, thermal conductivity, Thermal diffusivity are then crucial in exchanges with soil depths.

Exchanges heat does make the firsts in soil layers that can return to the surface if necessary, part of the energy. By cons, in a day, the heat storage in the soil (thermal inertia) concerns only the first layer of soil not exceeding 60 cm. I.e. the temperature field in the surface layer of the soil, which extends to a depth of 60 cm, is still sensitive to climatic variations recorded on the surface.

The objective of this work is precisely to propose a prediction model based on dimensional numerical finite element approximation for the evolution of the temperature field in a homogeneous soil.
2.1. Continuous Problem

For the case of a homogeneous and isotropic soil, the heat balance of a volume element, leads according to the depth $Z$, the propagation equation of heat without internal energy generation which is written:

\[ \frac{\partial T}{\partial t} - D \frac{\partial^2 T}{\partial Z^2} = 0 \quad \text{in } [a, b] \times [0, t_f] \]

This equation (general form) is called the diffusion equation of the temperature in the soil which using finite elements for the spatial discretization, and theta scheme for the time discretization.

2.1.1. Boundary conditions

\[ \begin{cases} 
T(t, a) = T_a = \alpha & \text{soil temperature} \\
T(t, b) = \beta \\
T(0, Z) = T_0(Z)
\end{cases} \]

we put [cf.12]

\[ D = \frac{\rho \cdot \lambda}{c_p} \quad \text{thermal diffusivity in the soil is } 10^{-6} m^2/s \]

where
\( \rho \): density
\( \lambda \): thermal conductivity
\( c_p \): the specific heat.

2.2. Discrete problem

To solve this problem (1) with conditions (2), we can proceed in two steps. In a first step, in space, we discretized by finite elements, that is to say as approaching space $H^1$ by a finite-dimensional space $V_1^h$ defined as

\[ V_1^h = \{ v_h \in C^2(\Omega) \cap H^1(\Omega), \ v_h \mid_{\Gamma=[a,b]} \in P_1 \text{ sur } \Omega \text{ et } T(a) = \alpha, T(b) = \beta \} \]

2.2.1. Spatial discretization

We define the following semi-discretized problem weak formulation. By multiplying the equation (1) by a test function $v \in H^1([a, b])$ and integrating in $[a, b] = \Omega$ we get

\[ \int_{\Omega} \frac{\partial T}{\partial t} \cdot v - D \int_{\Omega} \frac{\partial^2 T}{\partial Z^2} \cdot v = 0 \]

To construct an approximation $T^h \in V_1^h \subset H^1(\Omega)$, we introduce a mesh $\Omega$ compound $N + 1$. 

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If the mesh is uniform
\[ h = \frac{b - a}{N} \]

But in our case the mesh is not uniform
\[ h = \max h_j \]

Our first goal is to build a linear approximation (affine) piecewise

\[ V^h_1 \subset H^1(\Omega) = \left\{ v^h \in C^2(\Omega) \cap H^1(\Omega) : v^h \mid_{\Gamma_j} = \varphi_j \in P_i \quad v^h(a) = \alpha, v^h(b) = 0 \right\} \]

where
\[ v^h = \sum_{j=1}^{N-1} v_j(Z) \varphi_j(Z), \quad T^h = \sum_{i=0}^{N} T_i(t) \varphi_i(Z) \]

and \( \varphi_i = (\varphi_0, \varphi_1, ..., \varphi_N) \) form a basis vector \( T^h \) and \( \varphi_j = (\varphi_1, \varphi_2, ..., \varphi_{N-1}) \) form a basis vector \( v^h \)

the equation (3) equivalent to

\[ \int_{\Omega} \frac{\partial T}{\partial t} \cdot v + D \left( \int_{\Omega} \frac{\partial T}{\partial Z} \frac{\partial v}{\partial Z} - \int_{\Gamma} \frac{\partial T}{\partial Z} \cdot \eta \cdot v \cdot d\sigma \right) = 0 \]

The terms \( B, C \) are obtained using Green's formula

\[ \int_{\Omega} \frac{\partial^2 T}{\partial Z^2} \cdot v = \int_{\Omega} \frac{\partial T}{\partial Z} \frac{\partial v}{\partial Z} - \int_{\Gamma} \frac{\partial T}{\partial Z} \cdot \eta \cdot v \cdot d\sigma = \int_{\Omega} \frac{\partial}{\partial Z} \sum_{i=0}^{N} T_i(t) \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \sum_{j=1}^{N-1} v_j(Z) \varphi_j(Z) - \int_{\Gamma} \frac{\partial T}{\partial Z} \cdot \eta \cdot v \cdot d\sigma \]

To eliminate the term \( C \) can be changed the summation of \( T \) of \( i, N - 1 \) to \( 0, N \) (ie., put into consideration the edges \( \alpha \) and \( \beta \))

\[ \int_{\Omega} \frac{\partial^2 T}{\partial Z^2} \cdot v = \int_{\Omega} \frac{\partial}{\partial Z} \sum_{i=0}^{N} T_i(t) \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \sum_{j=1}^{N-1} v_j(Z) \varphi_j(Z) \]

with

\[ \sum_{i=0}^{N} T_i(t) \varphi_i(Z) = \alpha \varphi_0(Z) + \sum_{i=1}^{N-1} T_i(t) \varphi_i(Z) + \beta \varphi_N(Z) \]

and

\[ A = \frac{\partial}{\partial t} (T^h, v^h), \quad B = a(T^h, v^h) \]

So the equation (4) can be written in the following variational form:

\[ \frac{\partial}{\partial t} (T^h, v^h) + D a(T^h, v^h) = 0 \]
2.2.2. Time discretization

In a second step, we discretize (5) with respect to time using the theta scheme. It therefore seeks a sequence of elements $T^h_{n+1} \in V^h$ which approach $T_h(t_n)$; $t_n = n\Delta t$ and which are defined by the following formula.

Find $T^h_{n+1} \in V^h$, where for all $v^h \in V^h$:

$$\frac{T^h_{n+1} - T^h_n}{\Delta t}, v^h + D \cdot a (\theta T^h_{n+1} + (1-\theta) T^h_n, v^h) = 0, \text{ avec } \theta \in [0, 1]$$

2.3. Nature of the Problem Well Posed

By induction, since $T^h_0$, we have the existence and uniqueness result $(T^h_n) \geq 0$; we have the existence and uniqueness result. If $T^h_n$ is known, it is determined $T^h_{n+1}$ where for all $v^h \in V^h$:

$$\frac{T^h_{n+1} - T^h_n}{\Delta t}, v^h + D \theta \Delta t \cdot a (T^h_{n+1}, v^h) = (T^h_n, v^h) - D (1-\theta) \Delta t \cdot a (T^h_n, v^h)$$

2.3.1. Existance et unicité de la solution

According to the Lax-Milgram theorem the equation (4.7) admits a unique solution.

3. Stability Analysis of Theta Scheme

To investigate the stability using two techniques, increases energy and technical eigenvectors, which can conclude without any assumption on $V^h$ when the scheme is unconditionally stable.

3.1. Technique Increases Energy

We can write equation (7) as

$$(\frac{T^h_{n+1} - T^h_n}{\Delta t}, v^h) = D \cdot (\theta - 1) \cdot a (T^h_n, v^h) - D \cdot \theta \cdot a (T^h_{n+1}, v^h)$$

$$= - D \cdot [a ((1-\theta) T^h_n, v^h) + a (\theta T^h_{n+1}, v^h)]$$

$$= - D \cdot [a ((1-\theta) T^h_n + \theta T^h_{n+1}, v^h)]$$

$$= - D \cdot [a (\theta (T^h_{n+1} - T^h_n) + T^h_n, v^h)]$$

thus

$$\left| (\frac{T^h_{n+1} - T^h_n}{\Delta t}, v^h) + D \cdot [a (\theta (T^h_{n+1} - T^h_n) + T^h_n, v^h)] \right| = 0$$

We put $v^h = \theta (T^h_{n+1} - T^h_n) + T^h_n = \bar{U}^h$ (test function)
Equation (8) becomes

\[
(3.2) \quad \left( T_{n+1}^h - T_n^h, \frac{\theta (T_{n+1}^h - T_n^h) + T_n^h}{4} + D \cdot \Delta t \cdot a \left( \tilde{U}^h, \tilde{U}^h \right) \right) = 0
\]

The term \( A \) is equivalent to the following equation

\[
\begin{align*}
(T_{n+1}^h - T_n^h, & \theta (T_{n+1}^h - T_n^h) + T_n^h) \\
= & \theta (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) - \frac{1}{2} (T_{n+1}^h - T_n^h, -2T_n^h) \\
= & \theta (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) - \\
& - \frac{1}{2} (T_{n+1}^h - T_n^h, -T_n^h + T_{n+1}^h - T_n^h) \\
= & \theta (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) - \frac{1}{2} (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) \\
& - \frac{1}{2} (T_{n+1}^h - T_n^h, -(T_{n+1}^h + T_n^h)) \\
= & \theta (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) - \frac{1}{2} (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) + \\
& + \frac{1}{2} (T_{n+1}^h - T_n^h, T_{n+1}^h + T_n^h) \\
= & (\theta - \frac{1}{2}) (T_{n+1}^h - T_n^h, T_{n+1}^h - T_n^h) + \frac{1}{2} (T_{n+1}^h - T_n^h, T_{n+1}^h + T_n^h) \\
= & (\theta - \frac{1}{2}) \int (T_{n+1}^h - T_n^h)^2 + \frac{1}{2} \int (T_{n+1}^h - T_n^h) \cdot (T_{n+1}^h + T_n^h) \\
= & (\theta - \frac{1}{2}) \|T_{n+1}^h - T_n^h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|T_{n+1}^h - T_n^h\|_{L^2(\Omega)}^2 - \frac{1}{2} \|T_n^h\|_{L^2(\Omega)}^2 \\
\text{Injecting the term } A \text{ in equation (4.9)}
\]

\[
(\theta - \frac{1}{2}) \|T_{n+1}^h - T_n^h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|T_{n+1}^h\|_{L^2(\Omega)}^2 - \frac{1}{2} \|T_n^h\|_{L^2(\Omega)}^2 + D \cdot \Delta t \cdot a \left( \tilde{U}, \tilde{U} \right) = 0
\]

This means

\[
\begin{align*}
\frac{1}{2} \|T_{n+1}^h\|_{L^2(\Omega)}^2 - \frac{1}{2} \|T_n^h\|_{L^2(\Omega)}^2 + \Delta t \cdot D \cdot a \left( \tilde{U}, \tilde{U} \right) & = -(\theta - \frac{1}{2}) \|T_{n+1}^h - T_n^h\|_{L^2(\Omega)}^2 \\
\text{If } \theta \geq \frac{1}{2} \text{ on } a \\
& \frac{1}{2} \|T_{n+1}^h\|_{L^2(\Omega)}^2 - \frac{1}{2} \|T_n^h\|_{L^2(\Omega)}^2 + \Delta t \cdot D \cdot a \left( \tilde{U}^h, \tilde{U}^h \right) \leq 0 \\
\text{thus} \\
& \|T_{n+1}^h\|_{L^1(\Omega)}^2 + 2\Delta t \cdot D \cdot a \left( \tilde{U}^h, \tilde{U}^h \right) \leq \|T_n^h\|_{L^2(\Omega)}^2 \\
\text{then}
\end{align*}
\]
3.2. Technical eigenvectors

To perform the stability analysis of theta scheme for all theta arbitrary in [0, 1], we need to define the eigenvalues and eigenvectors of the bilinear form $a(\cdot, \cdot)$.

**Definition 1.** It is said that $\lambda$ is an eigenvalue of the bilinear form $a(\cdot, \cdot): V \times V \to \mathbb{R}$; and $\mu \in V^h$ is the eigenvector associated, if

$$a(u, v) = \lambda(u, v) \quad \forall v^h \in V^h$$

**Remark 1.** 1) If the bilinear form $a(\cdot, \cdot)$ symmetric and coercive, it has an infinite number of positive eigenvalues forming an unbounded sequence, in addition, the associated eigenvectors (also called eigenfunctions) form a basis of the space $V^h$. At the discrete level, the corresponding torque $(\lambda^h, \mu^h) \in \mathbb{R} \times V^h$ verify

$$a(u^h, v^h) = \lambda^h(u^h, v^h) \quad \forall v^h \in V^h$$

2) All the eigenvalues $(\lambda_1^h, \lambda_2^h, \ldots, \lambda_n^h)$ are positive. The corresponding eigenvectors $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ form a basis of the subspace $V^h$ and may be selected orthonormal, that is to say such as eigenvectors

$$(\varphi_i, \varphi_j) = \delta_{ij}, \forall i, j = 1; \ldots; N_h.$$ In particular, any function $v^h \in V^h$ can be represented as

$$v^h(Z) = \sum_{j=1}^{n-1} e_j \varphi_j(Z).$$

Now suppose that $\theta \in [0, 1]$ and let us focus on the case where the bilinear form $a(\cdot, \cdot)$ is symmetric. Let $\varphi_i$ the eigenvectors of $a(\cdot, \cdot)$, which form an orthogonal basis for the discrete $V^h$.

At each time step $T^h_k \in V^h$, it can be expressed $T^h_k$ as follows

$$T^h(Z) = \sum_{i=1}^{n} T^j_k \varphi_i(Z).$$
Since the $\omega_j^h$ are the eigenfunctions of $a(...)$, we get

\[ a(\varphi_i, \varphi_j) = \lambda_j^h(\varphi_i, \varphi_j) = \lambda_j^h \delta_{ij} = \lambda_j^h \text{ where } \delta_{ij} \text{ is Kronecker} \]

ie.,

\[ \frac{(T_{k+1}^i - T_k^i)}{\Delta t} + D \cdot [\theta T_{k+1}^i + (1 - \theta) T_k^i] \cdot \lambda_j^h = 0 \]

Solving this equation with respect to $T_{k+1}^i$, we find

\[ T_{k+1}^i = T_k^i \frac{1 - (1 - \theta)\lambda_j^h \Delta t}{1 + \theta \lambda_j^h D \cdot \Delta t} \]

The method is unconditionally stable, we must have

\[ \left| \frac{1 - (1 - \theta)\lambda_j^h D \cdot \Delta t}{1 + \theta \lambda_j^h D \cdot \Delta t} \right| < 1 \]

ie.,

\[ 2 \theta - 1 > \frac{2}{\lambda_j^h D \cdot \Delta t} \]

if $\theta > \frac{1}{2}$, this inequality is satisfied for any value of $\Delta t$, so the scheme is unconditionally stable, conversely for $\theta < \frac{1}{2}$, we must have

\[ \Delta t < \frac{2}{(1 - 2\theta)\lambda_j^h D} \]

That this relationship holds for all eigenvalues $\lambda_j^h$ of the bilinear form, just as it is for the most of them will be assumed to be $\lambda_{N_h}^h$.\n
\[ \Delta t < \frac{2}{(1 - 2\theta)\lambda_{N_h}^h D} \]

3.3. Matrix Formulation of the Problem

We take the equation (4.7)

\[ (T_{n+1}^h, v^h) + D \theta \Delta t a (T_{n+1}^h, v^h) = (T_n^h, v^h) - D (1 - \theta) \Delta t a (T_n^h, v^h), \omega = \theta \Delta t D \]

\[ \int_{\Omega} \left( \sum_{i=1}^{N-1} \sum_{j=1}^{n-1} T_i^{n+1}(t) \varphi_i(Z) \varphi_j(Z) \right) dZ + \omega \int_{\Omega} \sum_{i=1}^{N-1} \sum_{j=1}^{n-1} \frac{\partial}{\partial Z} \left( \sum_{i=1}^{N-1} T_i^{n+1}(t) \varphi_i(Z) \right) \frac{\partial}{\partial Z} \left( \sum_{j=1}^{n-1} \varphi_j(Z) \right) dZ \]
\[
\int_{\Omega} \left( \sum_{i=1}^{N-1} T_i^{n+1}(t) \varphi_i(Z) \sum_{j=1}^{n-1} u_j \varphi_j(Z) \right) dZ - (1 - \theta) \Delta t D \int_{\Omega} \frac{\partial}{\partial Z} \left( \sum_{i=0}^{N} T_i^{n+1}(t) \varphi_i(Z) \right) \frac{\partial}{\partial Z} \left( \sum_{j=1}^{n-1} u_j \varphi_j(Z) \right) dZ
\]

this equation is equal to

\[
\sum_{i=1}^{N-1} T_i^{n+1}(t) \int_{\sup \varphi_i \cap \sup \varphi_j} \varphi_i(Z) \varphi_j(Z) dZ + \omega \sum_{i=1}^{N-1} T_i^{n+1}(t) \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_i(Z) \frac{\partial}{\partial Z} \varphi_j(Z) dZ = 
\]

\[
\sum_{i=1}^{N-1} T_i^{n}(t) \int_{\sup \varphi_i \cap \sup \varphi_j} \varphi_i(Z) \varphi_j(Z) dZ - (1 - \theta) \Delta t D \sum_{i=0}^{N} T_i^{n}(t) \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_i(Z) \frac{\partial}{\partial Z} \varphi_j(Z) dZ
\]

thus

(3.4)

\[
\sum_{i=1}^{N-1} T_i^{n+1}(t) \left( \int_{\sup \varphi_i \cap \sup \varphi_j} \varphi_i(Z) \cdot \varphi_j(Z) dZ + \omega \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \varphi_j(Z) dZ \right)
\]

the equation (4.11) equal

\[
-\alpha (1 - \theta) \Delta t D \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_0(Z) \cdot \frac{\partial}{\partial Z} \varphi_1(Z) dZ - 
\]

\[
-\beta (1 - \theta) \Delta t D \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_N(Z) \cdot \frac{\partial}{\partial Z} \varphi_{N-1}(Z) dZ + 
\]

\[
+ \sum_{i=1}^{N-1} T_i^{n}(t) \left( \int_{\sup \varphi_i \cap \sup \varphi_j} \varphi_i(Z) \cdot \varphi_j(Z) dZ - (1 - \theta) \Delta t D \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \varphi_j(Z) dZ \right)
\]

thus

\[
\sum_{i=1}^{N-1} T_i^{n+1}(t) \left( \int_{x_{i-1}}^{x_{i+1}} \varphi_i(Z) \cdot \varphi_j(Z) dx + \theta \Delta t D \int_{x_{i-1}}^{x_{i+1}} \frac{\partial}{\partial Z} \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \varphi_j(Z) dZ \right) = 
\]

\[
= -\alpha (1 - \theta) \Delta t D \cdot \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_0(Z) \cdot \frac{\partial}{\partial Z} \varphi_1(Z) dZ - 
\]

\[
-\beta (1 - \theta) \Delta t D \cdot \int_{\sup \varphi_i \cap \sup \varphi_j} \frac{\partial}{\partial Z} \varphi_N(Z) \cdot \frac{\partial}{\partial Z} \varphi_{N-1}(Z) dZ + 
\]
\[(3.5) + \sum_{i=1}^{N-1} T_i^h(t) \left( \int_{x_i-1}^{x_{i+1}} \varphi_i(Z) \varphi_j(Z) dZ - (1 - \theta) \Delta t D \int_{x_i-1}^{x_{i+1}} \frac{\partial}{\partial Z} \varphi_i(Z) \frac{\partial}{\partial Z} \varphi_j(Z) dZ \right) \]

With basic functions

\[\varphi_i(z) = \begin{cases} 
\frac{z - x_i}{h} & z \in [x_{i-1}, x_i] \\
\frac{x_{i+1} - z}{h} & z \in [x_i, x_{i+1}] 
\end{cases} \]

\[\varphi_{i-1}(Z) = \begin{cases} 
\frac{Z - x_{i-1}}{h} & Z \in [x_{i-2}, x_{i-1}] \\
x_{i-1} - Z & Z \in [x_{i-1}, x_i] 
\end{cases} \]

with

\[h = (x_{i+1} - x_i) = (x_i - x_{i-1})\]

To facilitate the calculations we make a change of variable as

\[\begin{cases} 
Z = as + b \\
Z \in [x_i, x_{i+1}] \quad \text{translation} \quad s \in [0, 1]
\end{cases} \]

on a

\[\begin{cases} 
s = 0 \text{ d'où } b = x_i \\
s = 1 \text{ d'où } x_{i+1} = a + b \quad \Rightarrow \quad a = x_{i+1} - x_i = h
\end{cases} \]

giving

\[Z = hs + x_i\]

where

\[\varphi_i(Z)\big|_{[x_i, x_{i+1}]} = \varphi_0(s)\big|_{[0, 1]} = (1 - s)\]

and

\[Z \in [x_{i-1}, x_i] \quad \text{translation} \quad s \in [-1, 0] \]

so

\[\begin{cases} 
\text{for } s = 0 \text{ we have } b = x_i \\
\text{for } s = -1 \text{ we have } x_{i+1} = -a + b \quad \text{thus} \quad a = -x_{i-1} + x_i = h
\end{cases} \]

thus

\[Z = hs + x_i\]

and

\[\varphi_i(Z)\big|_{[x_{i-1}, x_i]} = \varphi_0(s)\big|_{[-1, 0]} = (1 + s)\]

\[dZ = hds\]

Finally we find

\[\varphi_0(s) = \begin{cases} 
(1 + s) & s \in [-1, 0] \\
(1 - s) & s \in [0, 1]
\end{cases} \quad \varphi_{-1}(s) = \begin{cases} 
(1 + s) & s \in [-2, -1] \\
-s & s \in [-1, 0]
\end{cases} \]

and

\[\varphi_{1}(s) = \begin{cases} 
(1 - s) & s \in [0, 1] \\
-s & s \in [1, 2]
\end{cases} \]

the equation (4.12) becomes
3.3.1. Calculation of terms of the matrices $M(\varphi_i, \varphi_j)$, $A(\varphi_i, \varphi_j)$

1) $i=j$, We have

$$M(\varphi_i, \varphi_i) = \int_{Z_{i-1}}^{Z_{i+1}} \varphi_i(Z) \cdot \varphi_i(Z) dZ = h \int_{-1}^{1} (\varphi_0(s))^2 ds = h \int_{-1}^{0} (\varphi_0(s))^2 ds + h \int_{0}^{1} (\varphi_0(s))^2 ds$$
\[
\int_{-1}^{0} (1 + s)^2 \, ds + \int_{0}^{1} (1 - s)^2 \, ds = \frac{1}{3} \left[ (1 + s)^2 \right]_{-1}^{0} - \frac{1}{3} \left[ (1 - s)^2 \right]_{0}^{1} \\
= \frac{1}{3} - 0 - 0 + \frac{1}{3} = \frac{2h}{3}
\]

thus

\[
M(\varphi_i, \varphi_i) = \frac{2h}{3}
\]

For

\[
A(\varphi_i, \varphi_i) = \int_{Z_{i-1}}^{Z_{i+1}} \left( \frac{\partial}{\partial Z} \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \varphi_i(Z) \right) \, dZ = \frac{1}{h} \int_{-1}^{1} (\varphi_0(s))' \, ds
\]

\[
= \frac{1}{h} \int_{0}^{1} (\varphi_0(s))' \, ds + \frac{1}{h} \int_{0}^{1} (\varphi_0(s))' \, ds = \frac{1}{h} \int_{-1}^{1} (1 - s)^2 \, ds = \frac{1}{h} \int_{-1}^{1} (1 - s)^2 \, ds = \frac{2}{h}
\]

thus

\[
A(\varphi_i, \varphi_i) = \frac{2}{h}
\]

2) \(i - 1 = j\), we have

\[
M(\varphi_i, \varphi_{i-1}) = \int_{Z_{i-1}}^{Z_{i+1}} \varphi_i(Z) \cdot \varphi_{i-1}(Z) \, dZ = \frac{1}{h} \int_{-1}^{1} \varphi_0(s) \cdot \varphi_{i-1}(s) \, ds = \\
= \frac{1}{h} \int_{-1}^{1} (1 + s) \cdot (-s) \, ds = \left( -\frac{s^2}{2} - \frac{s^3}{3} \right) \bigg|_{-1}^{0} = \left( \frac{1}{2} - \frac{1}{3} \right) \frac{1}{h} = \frac{h}{6}
\]

thus

\[
M(\varphi_i, \varphi_{i-1}) = \frac{h}{6}
\]

\[
A(\varphi_i, \varphi_{i-1}) = \int_{Z_{i-1}}^{Z_{i+1}} \left( \frac{\partial}{\partial Z} \varphi_i(Z) \cdot \frac{\partial}{\partial Z} \varphi_{i-1}(Z) \right) \, dZ = \frac{1}{h} \int_{-1}^{1} \varphi_0(s) \cdot \varphi_{i-1}(s) \, ds = \\
= \frac{1}{h} \int_{-1}^{1} (1 - s) \, ds = \frac{1}{h} \int_{-1}^{0} (-s) \, ds = \frac{1}{h}
\]
So
\[ A(\varphi_i; \varphi_{i-1}) = -\frac{1}{h} \]

3) \( i + 1 = j \)

Since the matrix is symmetrical are the same values as when \( i + 1 = j \), if \(|i - j| > 1\), the elements of the matrices A and M are zero.

4. Evaluation of the Error

To put our work on a practical plan, we performed measurements of the soil temperature and in the soil, that is to say, we used real data to demonstrate the accuracy of the method to using the estimate of the error.

4.1. Obtain the Original Data

The data available in the meteorological station Hydrometerological Institute of Formation and research located in Oran city in the west of Algeria are: the temperature at ground level, the maximum temperature and minimum soil temperatures for 4 levels in the soil (10, 20, 30 and 50 cm). We took the available data archives in November 2012, including the days you go from 10 to 14.

As for all numerical production data are needed at the edges of the field.

**Edge α:** is the ground temperature is calculated every 10 minutes (using a linear interpolation between the minimum and maximum for a given day, any time considering the maximum and minimum).

**Edge β:** According to a previous study [see 26], during the day, the heat storage in the soil for the first layer of the soil not exceeding 60 cm. The results show that the temperature field in the surface of the soil layer, which extends to a depth of 60 cm, remains sensitive to climatic variations recorded at the surface. Given that no data at 60 cm in the soil is estimated temperature at 60 cm after the thermal gradient variation observed in the soil. And after this temperature is fixed throughout the experiment (this assumption is proven by an earlier study. Results are in the following graph):
We note that the temperature at 60 cm remained virtually constant over 24 hours. We set \( \Delta t = 10\ \text{min} \) and \( h = \max h_j = 20\ cm \). The initial data are recorded in November 10, 2012 at 06.00 UTC. It runs every 10 minutes on the model until November 14, 2012 at 18.00 UTC. All displaying every 12 hours results obtained and compared with actual data to evaluate the mean squared error.

\[
MSE = \sum_{n=1}^{n-N} \frac{1}{N} (T_{\text{scheduled}} - T_{\text{real}})^2
\]

The calculations are done with two methods:
* With initialization: Turn the model and initializes the given 6h, 12h and 18h.
* Without initialization: Turn the model throughout the term without resetting the data.

4.2. Comments

The code used in this work is based on the finite element method to perform the prediction of ground temperature at different levels and for different maturities integration. For this. We compared the distribution of observed and simulated temperatures. We also calculated for each maturity squared error and the mean square error for the entire simulation.

The results of the simulations are shown below.
We note that the model is a good agreement with the observations for low depths to which [Fig. 1, 2] gives realistic enough temperatures, but despite what the differences are small begin to appear as the depth increases, it indicates that the model reproduces exactly the same temperature for the depth of 25 cm.

Fig. 2: Scheduled temperature after 12h

Fig. 3: Scheduled temperature after 24h
We note that the model always starts with a very good performance for low depths and calculating reaches the same temperature as that measured for an interval between 25 and 30 cm.

**Fig. 4:** Scheduled temperature after 36h

**Fig. 4:** Scheduled temperature after 48h
For maturities too high i.e. 36 h, 48 h and 60 h, we note that there is not a very good agreement between the temperatures predicted and observed for different depths.

The figures show that the model slightly underestimates the temperature to low depths and underestimated the depths. Sometimes a good forecast an average depth between 25 and 30 cm, but remains close to observations.
For maturities of 72 h, 84 h and 96 h, the comparison between the simulation results and observed data shows that temperatures are not well reproduced but remains close to the extent that measures the difference between the two temperatures does not exceed 0.6 °C.
From a statistical view point, the calculation of the mean square error provides a more view detailed for the behavior of the model. Although there is the lowest errors are observed for lower maturities. This is logical, since the model starts to move away from the value when deadlines increase, but in our case we note that the model returns a fairly good estimate for maturities of 84 h, 96 h and 108h.

5. Conclusion

The study showed that the finite element spatial approximation combined with a theta time scheme adequately reflects the variation of temperature fields at different soil depths. From the results obtained after the method gives a good simulation and explains to an accuracy of no more than 0.6 degrees maximum, the model has not diverged even after 108 hours.

As a logical model will begin to move when measuring the maturity increases, but it is not the case for our present model and this is due to the assumptions we have made:

* Linear interpolation between the minimum and maximum temperature ground temperatures for each 10 minutes.
* The presence of clouds and dew every morning throughout the term in question;
* The presence of rain.
* The presence of plants.

We noticed after the simulations the finite element method is estimated with good accuracy the temperature in the ground. The results are quite conclusive as well with that without initialization. In
addition, we wish to point out that the simulation results would have been better if the measurements were made in a greenhouse so immune to hydrometeors that significantly influence the variation of the surface temperature.

References


