Reduced Differential Transform Method for Time-Fractional Parabolic PDEs

Muhammad Sohail and Syed Tauseef Mohyud-Din

Department of Mathematics, HITEC University, TaxilaCantt, Pakistan

* Author to whom correspondence should be addressed; Email: syedtauseefs@hotmail.com, Tel: 92 333 5151290, Fax: 92 514544575

Article history: Received 8 August 2012, Received in revised form 11 November 2012, Accepted 20 November 2012, Published 26 November 2012.

Abstract: In this article, we apply Reduced Differential Transform Method (RDTM) to solve time-fractional fourth order parabolic PDEs. It is observed that the proposed technique (RDTM) is highly suitable for such problems. Numerical results re-confirm the efficiency and reliability of the proposed method.

Keywords: Reduced differential transform method, fourth order time-fractional parabolic partial differential equations, Maple.

1. Introduction

Fractional differential equations arise in almost all areas of physics, applied and engineering sciences [1-8]. In order to better understand these physical phenomena as well as further apply these physical phenomena in practical scientific research, it is important to find their exact solutions. The investigation of exact solutions of these equations is interesting and important. In the past several decades, many authors mainly had paid attention to study the solutions of such equations by using various developed methods. Recently, the Variational Iteration Method (VIM) [1-3] has been applied to handle various kinds of nonlinear problems, for example, fractional differential equations [4], nonlinear differential equations [5], nonlinear thermo elasticity [6], nonlinear wave equations [7]. In Refs. [8-13] Adomian’s Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and Variation of Parameter Method (VPM) are successfully applied to obtain the exact solution of differential equations. In the present article, we use Reduced
Differential Transform Method (RDTM) [14-18], to construct an appropriate solution to parabolic PDEs of fractional order.

The reduced differential transform technique is an iterative procedure for obtaining Taylor series solution of differential equations. This method reduces the size of computational work and easily applicable to many physical problems.

In order to apply RDTM to equation (1), some basic definitions are presented as follows.

### 1.1. Fourth-order Parabolic Partial Differential Equations

We consider the linear time-fractional fourth order parabolic partial differential equation, with variable coefficients, of the form

\[
\frac{\partial^{2n}u}{\partial t^{2n}} + \mu(x,y)\frac{\partial^4 u}{\partial x^4} + \lambda(x,y)\frac{\partial^4 u}{\partial y^4} = g(x,y,t), \quad a < x, y < b, \quad t > 0, \quad 0 < \alpha \leq 1,
\]

(1)

where \( \mu(x,y) \) and \( \lambda(x,y) \) are positive.

Subject to the initial conditions

\[
u(x,y,0) = f_0(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = f_1(x,y),
\]

and the boundary conditions are

\[
u(a,y,t) = g_0(y,t), \quad \nu(b,y,t) = g_1(y,t), \quad \nu(x,a,t) = k_0(x,t), \quad \nu(x,b,t) = k_1(x,t),
\]

\[
\frac{\partial^2 u}{\partial x^2}(a,y,t) = \bar{g}_0(y,t), \quad \frac{\partial^2 u}{\partial x^2}(b,y,t) = \bar{g}_1(y,t), \quad \frac{\partial^2 u}{\partial y^2}(x,a,t) = \bar{h}_0(x,t), \quad \frac{\partial^2 u}{\partial y^2}(x,b,t) = \bar{h}_1(x,t).
\]

Where the functions \( f_i, g_i, k_i, \bar{g}_i, \bar{h}_i \), \( i = 0,1 \) are continuous.

For the solution of this equation, we use Reduced Differential Transform Method (RDTM) which is explained as follows.

### 2. Definitions

The basic definitions of Reduced Differential Transform Method are explained as follows:

**Definition 2.1** If the function \( u(x,t) \) is analytic and differentiated continuously with respect tot, then let

\[
U_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[ \frac{\partial^{k\alpha} u}{\partial t^{k\alpha}} \right]_{t=0},
\]

(2)

where \( \alpha \) is a parameter describing the order of time-fractional derivative and the \( t \) – dimensional spectrum function \( U_k(x) \) is the transformed function. In this article, the lowercase \( u(x,t) \) represents the original function and the uppercase \( U_k(x) \) represents the transformed function.

**Definition 2.2** The differential inverse transform of \( U_k(x) \) is defined as follows:
Combining equation (2) and equation (3), it can be obtained that

\[ u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}. \]  

From the above definitions, it can be understood that the concept of reduced differential transform method is derived from the power series expansion of a function. The mathematical operations performed by reduced differential transform method are listed in Table 1.

### Table 1. Reduced Differential Transform Method

<table>
<thead>
<tr>
<th>Functional form</th>
<th>Transformed form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(x, t) )</td>
<td>( U_k(x) = \frac{1}{\Gamma(k\alpha + 1)} \left[ \frac{\partial^{k\alpha}}{\partial t^{k\alpha}} u(x, t) \right]_{t=0} )</td>
</tr>
<tr>
<td>( w(x, t) = u(x, t) \pm v(x, t) )</td>
<td>( W_k(x) = U_k(x) \pm V_k(x) )</td>
</tr>
<tr>
<td>( w(x, t) = \alpha u(x, t) )</td>
<td>( W_k(x) = \alpha U_k(x) )</td>
</tr>
<tr>
<td>( w(x, t) = u(x, t)v(x, t) )</td>
<td>( W_k(x) = \sum_{n=0}^{k} U_n V_{k-n} = \sum_{n=0}^{k} V_n U_{k-n} )</td>
</tr>
<tr>
<td>( w(x, t) = \frac{\partial^{n}}{\partial y^{n}} u(x, t) )</td>
<td>( W_k(x) = (k + 1)(k + 2) \ldots (k + n) U_{k+n}(x) )</td>
</tr>
<tr>
<td>( w(x, t) = x^{m} y^{n} u(x, t) )</td>
<td>( W_k(x) = x^{m} U_{k-n}(x) )</td>
</tr>
<tr>
<td>( w(x, t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x, t) )</td>
<td>( W_k(x) = \frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+N}(x) )</td>
</tr>
</tbody>
</table>

### 3. Numerical Applications

In this section, we use Reduced Differential Transform Method (RDTM) to solve some time-fractional fourth-order parabolic partial differential equations with variable coefficients. For the reliability of the method, three examples are presented.

#### Example 3.1
Consider the following one-dimensional singular time-fractional fourth-order parabolic partial differential equation with the initial and boundary conditions

\[ \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^{4} u}{\partial x^{4}} = 0, \quad 1 < x < 2, \quad t > 0, \quad 0 < \alpha \leq 1, \]  

With subject to the initial conditions

\[ u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 1 + \frac{x^5}{120}, \]
and the boundary conditions are
\[ u \left( \frac{1}{2}, t \right) = \left( 1 + \frac{1}{2} \right) \sin t, \ u(1, t) = \left( \frac{121}{120} \right) \sin t, \quad t > 0, \]
and
\[ \frac{\partial^2 u}{\partial x^2} \left( \frac{1}{2}, t \right) = \frac{1}{6} \left( \frac{1}{2} \right)^3, \quad \frac{\partial^2 u}{\partial x^2} (1, t) = \left( \frac{1}{6} \right) \sin t, \quad t > 0. \]

Where \( u = u(x, t) \) is a function of the variables \( x \) and \( t \).

Equation (5) can be written as
\[ \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = - \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 u}{\partial x^4}. \] (7)

Applying the reduced differential transform to equation (7), we have the following iteration equation,
\[ \frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2} (x) = - \left( \frac{1}{x} + \frac{x^4}{120} \right) \frac{\partial^4 U_k (x)}{\partial x^4}. \] (8)

Where the \( t \)-dimensional spectrum function \( U_k (x) \) is the transform function.

By using initial conditions (6), we write,
\[ U_0 (x) = 0, \quad U_1 (x) = 1 + \frac{x^5}{120}. \]

Using iterative relation (8), we obtain the following values of \( U_k (x) \) successively,
\[ U_2 (x) = 0, U_3 (x) = - \frac{1}{\Gamma(3\alpha + 1)} \left( 1 + \frac{x^5}{120} \right), U_4 (x) = 0, U_5 (x) = \frac{1}{\Gamma(5\alpha + 1)} \left( 1 + \frac{x^5}{120} \right), \]
\[ U_6 (x) = 0, U_7 (x) = - \frac{1}{\Gamma(7\alpha + 1)} \left( 1 + \frac{x^5}{120} \right), U_8 (x) = 0, U_9 (x) = \frac{1}{\Gamma(9\alpha + 1)} \left( 1 + \frac{x^5}{120} \right), \ldots \]

Finally the differential inverse transform (3) of function \( U_k (x) \) gives us
\[
\begin{align*}
  u(x, t) &= \sum_{k=0}^{\infty} U_k (x) t^{k\alpha} \\
  &= U_0 (x) + U_1 (x) t^{\alpha} + U_2 (x) t^{2\alpha} + U_3 (x) t^{3\alpha} + U_4 (x) t^{4\alpha} + U_5 (x) t^{5\alpha} + \ldots
\end{align*}
\]

\[
\begin{align*}
  u(x, t) &= \sum_{k=0}^{\infty} U_k (x) t^{k} \\
  &= \left( 1 + \frac{x^5}{120} \right) t^{\alpha} - \frac{1}{\Gamma(3\alpha + 1)} \left( 1 + \frac{x^5}{120} \right) t^{3\alpha} + \frac{1}{\Gamma(5\alpha + 1)} \left( 1 + \frac{x^5}{120} \right) t^{5\alpha} - \ldots
\end{align*}
\]
Example 3.2 Consider the following singular fourth order time-fractional parabolic partial differential equation in two space variables with the initial and boundary conditions

\[
\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + 2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} + 2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} = 0, \quad \frac{1}{2} < x, y < 1, \ t > 0, \ 0 < \alpha \leq 1, \tag{9}
\]

with the initial conditions

\[
u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \tag{10}
\]

and the boundary conditions are

\[
u(0.5, y, t) = \left( 2 + \frac{(0.5)^2}{6!} + \frac{y^6}{6!} \right) \sin t, \quad \nu(1, y, t) = \left( 2 + \frac{1}{6!} + \frac{y^6}{6!} \right) \sin t,
\]

\[
\frac{\partial^2 u}{\partial x^2}(0.5, y, t) = \frac{(0.5)^4}{6!} \sin t, \quad \frac{\partial^2 u}{\partial x^2}(1, y, t) = \frac{1}{24} \sin t,
\]

\[
\frac{\partial^2 u}{\partial y^2}(x, 0.5, t) = \frac{(0.5)^4}{6!} \sin t, \quad \frac{\partial^2 u}{\partial y^2}(x, 1, t) = \frac{1}{24} \sin t.
\]

Equation (9) can be written as,

\[
\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = -2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} - 2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} = 0, \quad \frac{1}{2} < x, y < 1, \ t > 0, \tag{11}
\]

Applying the reduced differential transform to equation (11), we have the following iteration equation,

\[
\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)} U_{k+2}(x, y) = 2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4}{\partial x^4} U_k(x, y) - 2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4}{\partial y^4} U_k(x, y). \tag{12}
\]
Where the $t$ -dimensional spectrum function $U_k(x, y)$ is the transform function.

By using initial conditions (10), we write,

$$U_0(x, y) = 0, \quad U_1(x, y) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}.$$ 

Using iterative relation (12), we obtain the following values of $U_k(x, y)$ successively,

$$U_2(x, y) = 0, U_3(x, y) = -\frac{1}{\Gamma(3\alpha + 1)} \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), U_4(x, y) = 0,$$

$$U_5(x, y) = \frac{1}{\Gamma(5\alpha + 1)} \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right), U_6(x, y) = 0,$$

$$U_7(x, y) = -\frac{1}{\Gamma(7\alpha + 1)} \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right),...$$

Finally the differential inverse transform (3) of function $U_k(x, y)$ gives us,

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y) t^{k\alpha}$$

$$= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) t^{\alpha} - \frac{1}{\Gamma(3\alpha + 1)} \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) t^{3\alpha} + \frac{1}{\Gamma(5\alpha + 1)} \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) t^{5\alpha} - ...$$

---

**Fig. 2(a):** 3D plot of $u(x, y, t)$ for $x = y$, $\alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.2

**Fig. 2(b):** 2D plot of $u(x, y, t)$ for $x = y = 1$, $\alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.2
Example 3.3 We next consider the following time-fractional fourth order parabolic equation subject to the initial and boundary conditions

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + \left(\frac{x}{\sin x} - 1\right) \frac{\partial^4 u}{\partial x^4} = 0, \ 0 < x < 2, 0 < \alpha \leq 1,$$

subject to the initial conditions

$$u(x, 0) = x - \sin x, \ \frac{\partial u}{\partial t}(x, 0) = -(x - \sin x), 0 < x < 1$$

(13)

and the boundary conditions are

$$u(0, t) = 0, \ u(1, t) = e^{-t}(1 - \sin 1), \ \frac{\partial^2 u}{\partial x^2}(0, t) = 0, \ \frac{\partial^2 u}{\partial x^2}(1, t) = e^{-t} \sin 1.$$  

(14)

Equation (13) can be written as,

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = -\left(\frac{x}{\sin x} - 1\right) \frac{\partial^4 u}{\partial x^4}, \ 0 < x < 1, \ t > 0.$$  

(15)

Applying the reduced differential transform to equation (15), we have the following iteration relation,

$$\frac{\Gamma(\kappa+2\alpha+1)}{\Gamma(\kappa+1)} U_{k+2}(x) = -\left(\frac{x}{\sin x} - 1\right) \frac{\partial^4}{\partial x^4} U_k(x).$$  

(16)

Where the $t$-dimensional spectrum function $U_k(x)$ is the transform function.

By using initial conditions (14), we write,

$$U_0(x) = x - \sin x, \ U_1(x) = -(x - \sin x).$$

Using iterative relation (16), we obtain the following values of $U_k(x)$ successively,

$$U_2(x) = \frac{1}{\Gamma(2\alpha + 1)} (x - \sin x), \ U_3(x) = -\frac{1}{\Gamma(3\alpha + 1)} (x - \sin x), \ U_4(x) = \frac{1}{\Gamma(4\alpha + 1)} (x - \sin x),$$

$$U_5(x) = -\frac{1}{\Gamma(5\alpha + 1)} (x - \sin x), \ U_6(x) = \frac{1}{\Gamma(6\alpha + 1)} (x - \sin x), \ U_7(x) = -\frac{1}{\Gamma(7\alpha + 1)} (x - \sin x), ...$$

Finally the differential inverse transform (3) of function $U_k(x)$ gives us

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^k = (x - \sin x) - (x - \sin x) t^\alpha + \frac{1}{\Gamma(2\alpha + 1)} (x - \sin x) t^{2\alpha} - ...$$

Fig. 3(a): 3D plot of $u(x, t)$ for $\alpha = 0.25, \alpha = 0.50, \alpha = 0.75, \alpha = 1$ for example 3.3

Copyright © 2012 by Modern Scientific Press Company, Florida, USA
4. Conclusion

In this article, Reduced Differential Transform Method (RDTM) has been successfully applied to handle the time-fractional linear fourth-order parabolic partial differential equations with variable coefficients. The results show that the RDTM is a powerful mathematical tool for solving partial differential equations having wide range of applications in engineering. In our work, we use Maple package for the graphical presentation of solution for different values of $\alpha$. The numerical results reveal the complete reliability of RDTM.

References


